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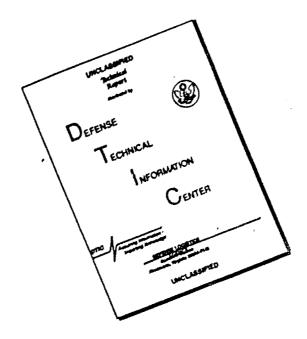
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Reports of the Institute of Geodesy, Photogrammetry and Cartography

Report No. 27

Practical Computation of Gravity at High Altitudes

by

R. A. Hirvonen and Helmut Moritz

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

Contract No. AF 19(628)-2771 Project No. 7600 Task No. 760002





The Ohio State University Research Foundation Columbus, Ohio 43212

May 1963

AFCRL-63-702

INSTITUTE OF GEODESY, PHOTOGRAMMETRY AND CARTOGRAPHY

W. A. Heiskanen, Director

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FOREWORD

This report was prepared by Dr. R. A. Hirvonen and Dr. Helmut Moritz, Research Associates, of the Institute of Geodesy, Photogrammetry and Cartography of The Ohio State University, under Air Force Contract No. AF 19(628)-2771, OSURF Project No. 1613, under the supervision of Dr. Weikko A. Heiskanen, Director of the Institute. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen Williams and Mr. Bela Szabo, Project and Task scientists.

ABSTRACT

This report presents and compares methods for computing the gravity vector outside the earth. The gravity vector is conveniently split up into a normal part, the vector of normal gravity, and an anomalous part, the vector of gravity disturbance.

Part I gives the theoretical foundations and a practical method \wedge for the computation of the vectors of normal gravity and gravitation. The method is adapted for electronic computation. It is illustrated by a numerical example \wedge

Then the formulas for the computation of the gravity disturbance vector from free-air gravity anomalies Δg at the surface of the earth are developed, (a) for the <u>direct method</u>, which uses Δg only, and (b) for the <u>coating method</u>, which requires Δg and the geoid heights N but involves simpler formulas.

The components of the gravity vector are first computed in a local coordinate system and then transformed to the geocentric world system by a spatial rotation. If the gravity vector is required along a rocket trajectory, then also the components along and normal to the trajectory can be computed by a spatial rotation.

Part II is concerned with accuracy studies in order to determine the best practical procedure for the computation of the gravity disturbances.

The standard errors of the components of the gravity disturbances, due to the interpolation of the gravity material, are, approximately, inversely proportional to the elevation and very small for high altitudes, provided there is uniform coverage by gravity stations.

The influence of the distant zones is considered in some detail. This influence decreases very slowly beyond a certain radius, so that it is impractical to go farther than about 30° in the direct method and 20° in the coating method (with an error of about ±5 mgal in both cases), unless the integration is extended over the whole earth, which is necessary for higher accuracy.

There is another method, for which the influence of the distant zones is completely negligible and which furthermore is the simplest. the upward continuation of the surface disturbances. If presupposes.

however, the deflections of the vertical $\,\xi\,$ and $\,\eta\,$ for the horizontal components, and $\,\Delta g\,$ and $\,N\,$ for the vertical component.

For practical use the following methods are proposed: if only Δg is given, the direct method; if Δg and N are given, the coating method for the horizontal components and the upward continuation for the vertical component; if Δg , ξ , η are given, the upward continuation for all three components.

Finally, a detailed practical computation procedure is described for the practically most important case that Δg and N are given.

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bу

R. A. Hirvonen

INTRODUCTION

The purpose of this report is to develop a method of computation of the gravity vector for a great number of points at high altitudes, e.g. for trajectories of moving bodies, when the geocentric coordinates are given. Formulas will be summarized in a form suitable for automatic high speed computers. Numerical examples illustrate the method.

First, the components of the normal gravitation (attraction of a reference ellipsoid) will be computed. Those of the normal gravity can then be obtained by subtraction of the components of the centrifugal force. Finally, the deviations of the actual components from these normal values will be evaluated on the basis of the free air anomalies of the gravity observed on the physical surface of the earth. The accuracy of the method is studied in Part II.

The method is based on the formulas first published in [Hirvonen, 1959], but several practical improvements have been introduced.

1. CONVERSION OF THE COORDINATES

It is supposed that the Cartesian geocentric coordinates x,y,z

are given for each point P at which the gravity will be computed. The z-axis is the axis of the earth's rotation, the x-axis has the longitude 0° (Greenwich) and the y-axis has the longitude 90°E. These coordinates must be converted into three other systems.

I. Geocentric coordinates, defined by

$$x = r \cos \Psi \cos \lambda$$

(I)
$$y = r \cos \Psi \sin \lambda$$

$$z = r \sin \Psi$$

and called

r = radius vector

 Ψ = geocentric latitude

 $\lambda = geographic longitude$

When x,y,z are given, r, Y, λ can easily be computed:

$$(1.1) p2 = x2 + y2$$

$$(1.2)$$
 $r^2 = p^2 + z^2$

(1.3)
$$\cos \lambda = \frac{x}{p}$$

(1.4)
$$\sin \lambda = \frac{y}{p}$$

$$(1.5) \quad \cos \Psi = \frac{p}{r}$$

(1.6)
$$\sin \Psi = \frac{z}{r}$$

For the computation of the normal gravitation, we shall use the function

(1.7)
$$F = \cos 2 \Psi = \frac{p^2 - z^2}{r^2}$$

II. Geographic coordinates, defined by

$$p = (N + H) \cos \phi$$

$$z = (N + H - e^{2}N) \sin \phi$$

$$(1.8) \qquad N = \frac{a}{W}$$

(1.9)
$$W^2 = 1 - e^2 \sin^2 \varphi$$

 $e^3 = \frac{a^3 - b^2}{a^2}$

and called

 $\varphi = geographic latitude$

H = geographic height

The geographic coordinates refer to a reference ellipsoid with equatorial radius a and polar radius b, and they are used for the computation of triangulations, in spite of the fact that the "orientation" of the ellipsoid (or the "data" of the triangulation system) cannot be determined without world-wide geodetic operations. The names suggested here are, however, not yet in a general use.

The normal gravitation is the attraction of the reference ellipsoid which is supposed to have the same mass as the actual earth. The masses inside the rotating ellipsoid are supposed to be distributed in such a way that the combined potential of the attraction and of the centrifugal force is constant at the outer surface and very close to the actual potential of the gravity at the mean sea level.

At higher altitudes, the geographic coordinates are less practical for the exact computation of the geodetic quantities. Especially, we cannot use them for the computation of the normal gravity. However, we should use them for the evaluation of the deviations of the actual gravity from the normal gravity on the basis of the anomalies observed on the ground.

The computation of $\,\phi\,$ and H $\,$ from x,y,z is rather complicated. First, we obtain

(1.10)
$$H = \frac{p}{\cos \phi} - N$$

(1.11)
$$H = \frac{Z}{\sin c} - N + e^2 N$$

The difference of these equations gives

(1.12)
$$\tan \varphi = \frac{z}{p} + e^2 \frac{N}{p} \sin \varphi$$

This equation will be solved with respect to ϕ by successive approximations. The first approximation is usually obtained by H = 0 in which case we have

(1.13)
$$\tan \sigma_1 = \frac{1}{1-e^2} \cdot \frac{z}{p}$$

For the higher altitudes, it is slightly better to start from $H = e^2 N = 43 \text{ km}$, in which case we obtain

(1.14)
$$\tan \varphi_1 = (1 + e^2) \frac{z}{p}$$

The latter case will be used in our present applications.

Any approximate value tan φ_1 can be improved as follows. Compute (1.15) $\sin^2 \varphi_1 = \frac{\tan^2 \varphi_1}{1 + \tan^2 \varphi_1}$

and W_1 by the aid of the value thus obtained. Then we have a second approximation

$$(1.16)$$
 $W_1^2 = 1 - e^2 \sin^2 \varphi_1$

$$(1.17) \quad \delta = e^{3} a \frac{\sin \varphi_{1}}{pW_{1}}$$

(1.18)
$$\tan \varphi_2 = \tan \Psi + \delta$$

If this new approximation is improved in the same way, the change is already much smaller and it can be estimated roughly:

(1.19)
$$\tan \varphi_3 - \tan \varphi_2 = \delta \cdot \frac{\cos^2 \varphi_1}{\tan \varphi_1} (\tan \varphi_2 - \tan \varphi_1)$$

In the numerical example given in chapter 4 we have H=129. Even in this extreme case the error of ϕ_3 is 0".0002 only. That of ϕ_2 is 0".035 and that of ϕ_1 is 9".

In our present problem, we shall use ϕ only for the computation of the disturbances of gravity. Therefore, we may think that the first approximation ϕ_1 already is quite sufficient for this partic-

ular purpose. However, we have included the computation of the correct value into our program because it may be useful for other purposes.

Note that φ is not the direction of the normal gravity at the elevated point P but the direction of the normal of the reference ellipsoid. The former direction can be computed only when the components of the normal gravity are known:

(1.20)
$$\tan \varphi'' = \frac{\gamma_z}{\gamma_p}$$

We shall call ϕ'' the <u>geodetic latitude</u>, bearing in mind the definition: ξ - component of the deflection of the vertical is the difference of the <u>astronomical</u> latitude ϕ' and <u>geodetic</u> latitude ϕ'' at the elevated point P .

The difference ϕ "- ϕ is caused by the curvature of the normal plumb line. In [Hirvonen, 1960], a series is given for the computation of ϕ "- ϕ

$$\phi$$
" - ϕ = $\sin 2 \phi$ (0",170293 H + 0 ,001103 H $\cos 2 \phi$ + 0 ,000034 H²)

(H in kilometers). In the present problem, however, we cannot use this formula for the computation of $\,\phi\,$ because H still is unknown.

Therefore, we have to compute φ by successive approximations as

described above. When the final approximation has been found, H can be obtained from (1.10).

For latitudes higher than 45°, our program should be based on an alternative set of formulas. Instead of (1.14) through (1.19), compute

(1.22) cot
$$\varphi_1 = \frac{p}{z(1+e^2)}$$

$$(1.23) \quad \cos^2 \varphi_1 = \frac{\cot^2 \varphi_1}{1 + \cot^2 \varphi_1}$$

(1.24)
$$\cot \varphi_2 = \frac{p}{z} - \frac{e^2 a}{W_1} \frac{\cos \varphi_1}{z} = \cot \Psi - \delta$$

(1.25)
$$\cot \varphi_3 - \cot \varphi_2 = \delta \cdot \frac{\sin^2 \varphi_1}{\cot \varphi_1} (\cot \varphi_2 - \cot \varphi_1)$$

With the final approximation of ϕ , compute H from (1.11).

III. Elliptic coordinates, defined by

$$p = \frac{c}{\sin \epsilon} \cos \beta$$

(III)
$$z = \frac{c}{\tan \varepsilon} \sin \beta$$

$$c = e a$$

and called

 ε = angular eccentricity

 β = reduced latitude

At the surface of the reference ellipsoid, ϵ has a constant value

 ϵ_{0} with

(1.26)
$$\sin \varepsilon_0 = e$$

When p and z are given, we could compute ε and β by rigorous formulas

$$(1.27)$$
 $k^2 = r^2 + c^2$

$$(1.28)$$
 $h^4 = k^4 - 4p^2 e^2$

$$(1.29) \quad \sin^2 \varepsilon = \frac{k^2 - h^2}{2p^2}$$

(1.30)
$$\cos^2 \beta = \frac{2p^2}{k^2 + h^2}$$

For practical computations, however, we shall use power series.

By the aid of

$$(1.31)$$
 $x^2 = \frac{c^2}{r^2}$

we can replace the formulas above by

$$(1.32)$$
 $k^2 = r^2(1 + n^2)$

$$(1.33)$$
 h⁴ = r⁴ $(1 - 2\mu^2 \cos 2\Psi + \mu^4)$

(1.34)
$$\sin^2 \varepsilon = \varkappa^2 [1 - \varkappa^2 \sin^2 \Psi$$

$$- \varkappa^4 \sin^2 \Psi \cos 2 \Psi$$

$$- \varkappa^6 \sin^2 \Psi (\frac{5}{4} \cos 2 \Psi - \frac{1}{4})]$$

(1.35)
$$\cos^2 \beta = \cos^2 \Psi \left[1 - \kappa^2 \sin^2 \Psi - \kappa^4 \sin^2 \Psi \cos 2 \Psi\right]$$

Using the abbreviating symbol

(1.7)
$$F = \cos 2 \Psi$$

we obtain, after lengthy but easy computations,

$$\varepsilon = \varkappa + \frac{1}{12} \varkappa^3 (3F - 1)
+ \frac{1}{160} \varkappa^5 (35F^2 - 10F - 13)
+ \frac{1}{128} \varkappa^7 (33F^3 - 9F^2 - 21F + \frac{19}{7})$$

2. THE POTENTIAL OF NORMAL GRAVITATION

The Newtonian attraction of the reference ellipsoid is called the normal gravitation of the earth. The potential of this attraction can be expressed in a closed form:

(2.1)
$$V = \frac{fM}{c} \epsilon + \frac{\omega^2 a^2}{2q} q(\frac{2}{3} - \cos^2 \beta)$$

where

f : gravitational constant,

M: mass of the earth,

w: rotation speed of the earth.

The auxiliar variable

(2.2)
$$q = \frac{1}{2} [\epsilon - 3 \cot \epsilon (1 - \epsilon \cot \epsilon)]$$

is a function of $\,\varepsilon\,$ only. At the surface of the reference ellipsoid q has a constant value q $_{_{\bigcirc}}.$

The closed formula (2.2) of q is not suitable for practical computations, but must be replaced by power series:

(2.3)
$$q = \frac{2}{15} \tan^3 \epsilon - \frac{4}{35} \tan^5 \epsilon + \frac{2}{21} \tan^7 \epsilon - \dots$$

OT

(2.4)
$$q = \frac{2}{15} \sin^3 \epsilon + \frac{3}{35} \sin^5 \epsilon + \frac{5}{84} \sin^7 \epsilon + \cdots$$

Therefore, it is only logical that we use power series throughout for our computations. By the aid of (1.34), (2.4) gives

(2.5)
$$q = \frac{2}{15} \kappa^3 + \frac{1}{10} \kappa^5 (F - \frac{1}{7}) + \frac{1}{560} \kappa^7 (63F^3 - 10F - \frac{59}{3})$$

Now we have to insert (1.35), (1.36) and (2.5) into (2.1). Using another abbreviating symbol for the constant

(2.6)
$$C = \frac{2}{15} \frac{\omega^3 a^3 c}{fMq}$$

we can write the result in form

$$V = \frac{fM}{c} \times \left\{ 1 + \frac{1}{4} \times^{2} (1 - C)(F - \frac{1}{3}) + \frac{1}{224} \times^{4} (7 - 10C)(7F^{2} - 2F - \frac{13}{5}) + \frac{1}{128} \times^{6} (3 - 5C)(11F^{3} - 3F^{2} - 7F + \frac{19}{21}) \right\}$$

3. THE COMPONENTS OF NORMAL GRAVITY

Differentiation of (2.7) and (1.31) with respect to r gives the component of the normal gravitation toward the center of the earth:

$$\Gamma_{r} = -\frac{\partial V}{\partial r} = \frac{fM}{c^{2}} \kappa^{2} \left\{ 1 + \frac{1}{4} \kappa^{2} \left(1 - C \right) (3F - 1) + \frac{1}{224} \kappa^{4} \left(7 - 10C \right) (35F^{2} - 10F - 13) + \frac{1}{384} \kappa^{6} \left(3 - 5C \right) (231F^{3} - 63F^{2} - 147F + 19) \right\}$$

Differentiation of (2.7) and (1.7) with respect to Ψ and division by r give the component perpendicular to Γ_r :

$$\Gamma_{\Psi} = -\frac{1}{r} \frac{\partial V}{\partial \Psi} = \frac{fM}{c^2} \varkappa^4 \sin \Psi \cos \Psi \{1 - C$$

$$+ \frac{1}{28} \varkappa^2 (7 - 10C)(7F - 1)$$

$$+ \frac{1}{32} \varkappa^4 (3 - 5C)(33F^2 - 6F - 7)\}$$

In order to facilitate the numerical computations, we replace the increasing powers of \varkappa^2 by those of

$$(3.3) \quad G = \frac{fM}{r^3}$$

using convenient units. In other words, we insert:

$$(3.4) \qquad \kappa^{3} = G \cdot \frac{e^{3}}{fM}$$

For the international ellipsoid with the international normal formula of gravity we obtain, when r is expressed in kilometers and Γ in gal = cm sec $^{-2}$:

$$(3.5) \qquad G = \frac{3986 \ 3290.45}{r^2}$$

$$\Gamma_{\mathbf{r}} = 1000 \text{ G}$$

$$- 0.835 888 \text{ G}^{2}$$

$$+ 2.507 664 \text{ G}^{3} \cos 2\Psi$$

$$- 0.005 118 \text{ G}^{3}$$

$$- 0.003 937 \text{ G}^{3} \cos 2\Psi$$

$$+ 0.013 778 \text{ G}^{3} \cos^{2} 2\Psi$$

$$+ 0.000 007 \text{ G}^{4}$$

$$- 0.000 054 \text{ G}^{4} \cos^{2} 2\Psi$$

$$- 0.000 023 \text{ G}^{4} \cos^{2} 2\Psi$$

$$+ 0.000 085 \text{ G}^{4} \cos^{3} 2\Psi$$

$$\Gamma_{\Psi} = \sin \Psi \cos \Psi \{3.343 551 \text{ G}^{2}$$

$$- 0.003 149 \text{ G}^{3}$$

$$+ 0.022 045 \text{ G}^{3} \cos 2\Psi$$

$$- 0.000 031 \text{ G}^{4}$$

$$- 0.000 026 \text{ G}^{4} \cos 2\Psi$$

$$+ 0.000 145 \text{ G}^{4} \cos^{2} 2\Psi \}$$

As G is always slightly smaller than 1, it is easy to see that the accuracy of one milligal can be obtained without the terms with \mathbb{G}^4 .

The components of gravitation in directions of the Cartesian coordinate axes are here chosen to be positive towards the origin. They can be computed by formulas

(3.8)
$$\Gamma_{p} = \Gamma_{r} \cos \Psi - \Gamma_{\Psi} \sin \Psi$$

(3.9)
$$\Gamma_{x} = \Gamma_{p} \cos \lambda$$

(3.10)
$$\Gamma_y = \Gamma_p \sin \lambda$$

(3.11)
$$\Gamma_z = \Gamma_r \sin \Psi + \Gamma_{\Psi} \cos \Psi$$

The components of the normal gravity are obtained by subtraction of those of the centrifugal force:

(3.12)
$$\gamma_p = \Gamma_p - \omega^2 p$$

(3.13)
$$\gamma_x = \Gamma_x - \omega^2 x$$

$$(3.14) \quad \gamma_{y} = \Gamma_{y} - \omega^{2} y$$

$$(3.15) \gamma_z = \Gamma_z$$

The total gravitation is

(3.16)
$$\Gamma = \sqrt{\Gamma_s + \Gamma_g}$$

and the total gravity

(3.17)
$$\gamma = \sqrt{\gamma_p^2 + \gamma_z^2}$$

The direction of the former is

(3.18)
$$\Phi = \arctan \frac{\Gamma_Z}{\Gamma_p}$$

that of the latter was given in (1.20).

4. SUMMARY OF FORMULAS FOR NORMAL GRAVITY

The formulas given above will be summarized here in form of a program for automatic computers. First, the constants are listed which pertain to the international ellipsoid and to the international normal

formula of gravity. Then the computations are described in detail, using a numerical example as illustration.

Input	Constants	
(1)	1,0000 0000	
(2)	$e^2 = 0.0067 2267$	
(3)	$1 + e^2 = 1,0067 2267$	
(4)	a = 6378,3880	
(5)	e ² a = 42.8798	
(6)	f M = 3986 3290	
(7)	1000,0000	
(8)	0,83589	
(9)	2,50766	
(10)	0,00512	
(11)	0,00394	
(12)	0,01378	
(13)	3, 34355	
(14)	0,00315	
(15)	0.02204	
(16)	$w^2 = 0.000531749$	
Input	Variables	Example
(17)	Identification number of P	
(18)	x	521,0517
(19)	У	4957.4762
(20)	${f z}$	4169.1994

Program of Computations

(21)	pa	(18) • (18) + (19) • (19)	2484 8065
(22)	z	(20) • (20)	1738 2224
(23)	r ^a	(21) + (22)	4223,0289
(24)	cos 2¥ [[(21) - (22)] ÷ (23)	0,1767 8877
(25)	р	√(2l)	4984,7833
(26)	r	√(23)	6498,4836
(27)	cos ¥	(25) ÷ (26)	0.7670 6869
(28)	sin Y	(20) ÷ (26)	0,6415 6496
(29)	tan Y	(20) ÷ (25)	0.8363 8528
(30)	cos λ	(18) ÷ (25)	0.1045 2846
(31)	$\sin \lambda$	(19) ÷ (25)	0,9945 2191
(32)	cot λ	(18) ÷ (19)	0,1051 0423
(33)	cosΨ cosλ	(27) • (30)	0.0802
(34)	sinΨ cosλ	(28) • (30)	0,0670
(35)	cosΨ sinλ	(27) • (31)	0.7629
(36)	sinΨ sinλ	(28) • (31)	0,6381
(37)	λ	arc cot (32)	. 84°00! 0000
(38)	$\tan \phi_1$	(3) • (29)	0,8420 0802
(39)	$\tan^2 \phi_1$	(38) • (38)	0,7089 7751
(40)	$\sin^2 \phi_1$	$(39) \div [(1) + (39)]$	0.4148 5479
(41)	$M_{\mathbf{S}}$	(1) - (2) • (40)	0.9972 1107
(42)	$\text{sin }\phi_1$	√(40)	0.6440 9222
(43)	W	√ (41)	0.9986 0456
(44)		(5) • (42) ÷ (25)	0.0055 4057
(45)	δ	(44) ÷ (43)	0,0055 4831

(46)	tan φ₂	(29) + (45)	0.8419 3359
(47)		(38) - (46)	0,0000 7443
9 (48)		(38) • [(1) + (39)]	1,4390
(49)		(47) • (45) ÷ (48)	0,0000 0029
(50)	tan φ	(46) - (49)	0.8419 3330
(51)	tan ² φ	(50) • (50)	0.7088 51.68
(52)	sec φ	$\sqrt{(1) + (51)}$	1.3072 3054
(53)	cos φ	(1) ÷ (52)	0.7649 7601
(54)	$\text{sin } \phi$	(50) • (53)	0.6440 5878
(55)	φ	arc tan (50)	40°05°.7085
(56)		(25) • (52)	6516,2610
(57)		(4) ÷ (43)	6387.3011
(58)	H	(56) - (57)	128.96
(59)	G	(6) ÷ (23)	0.9439 5022
(60)	G₅	(59) • (59)	0.8910 42
(61)	$G_{\mathbf{z}}$ E	(60) • (24)	0.1575 26
(62)	G	(60) • (59)	0,8411
(63)	$^{ m G}{}^{ m a}{}_{ m F}$	(62) • (24)	0,1487
(64)	G_3 Es	(63) • (24)	0.0263
(65)		(7) · (59)	943,950 22
(66)		(8) • (60)	0.744 81
(67)		(9) • (61)	0,395 02
(68)		(10) · (62)	0.004 31
(69)		(11) · (63)	0,000 59
(70)		(12) • (64)	0.000 36

Input		Variables	Example
(71)	$\Gamma_{\mathbf{r}}$	(65)-(66)+(67)-(68)-(69)+(70)	943,595 90
(72)	_	(13) • (60)	2,979 24
(73)		(14) • (62)	0,002 65
(74)		(15) • (63)	0.003 28
(75)		(72) - (73) + (74)	2,979 87
(76)	Γ_{Ψ}	(75) • (27) • (28)	1,466 47
(77)	Γ _p	(71) · (27) - (76) · (28)	722,862 04
(78)	$\Gamma_{\rm z} = \gamma_{\rm z}$	(71) · (28) + (76) · (27)	606,502 95
(79)	$\Gamma_{\mathbf{x}}$	(77) · (30)	75,559 66
(80)	Γ _y	(77) • (31) •	718,902 14
(81)	$\gamma_{ m p}$	(77) - (16) • (25)	720,211 39
(82)	$\gamma_{_{\mathbf{X}}}^{^{\mathbf{r}}}$	(79) - (16) · (18)	75,282 59
(83)	γ _y	(80) - (16) · (19)	716,266 01
(84)	tan Þ	(78) ÷ (77)	0,8390 3002
(85)	tan φ"	(78) ÷ (81)	0.8421 1796
(86)	$L_{\mathbf{S}}$	(71) · (71) + (76) · (76)	890375.37
(87)	. _Y a	(78) • (78) + (81) • (81)	886550,27
(88)	Γ	√ (86)	943.59704
(89)	γ	√ (87)	941.56799
(90)	Φ	arc tan (84)	39°59 '. 8596
(91)	φ''	arc tan (85)	40°06'.0800

Output

5. GRAVITY DISTURBANCES

If the free air anomalies Δg are known everywhere on the physical surface S of the earth, the potential disturbance T at any point on or above S can be computed. The computation is very complicated, especially if the point is very close to high and steep mountains. In most cases, however, the generalized formula of Stokes gives a sufficient approximation.

We shall use following notations:

P: fixed point at which T is wanted;

M: "moving" point at S;

r: distance from the centre of the earth;

 ψ : angle between radii $\textbf{r}_{\underline{P}}$ and $\textbf{r}_{\underline{M}}$

 α = azimuth from P to M;

(5.1)
$$t = \frac{r_{M}}{r_{D}}$$
;

(5.2)
$$D^2 = 1 - 2t \cos \psi + t^2$$
;

(5.3)
$$d\sigma = \sin \psi d\Psi d\alpha = \cos \phi d\phi d\lambda$$

is the element of solid angle, situated at point M.

The generalized formula of Stokes reads

(5.4)
$$T_{p} = \frac{r_{p}}{4\pi} \int \Delta g_{M} t^{2} \{ \frac{2}{D} + 1 - 3D - t \cos \psi (5+3\ln \frac{D+1-t \cos \psi}{2}) \} d_{\sigma}$$

The integration must be carried out over the entire surface S. To a very good approximation,

$$(5.5)$$
 $r_{M} = r_{P} - H$

where H is the height of P above S . Usually, r_{M} can even be replaced by a constant R , the mean radius of the earth.

Especially, if H = 0 or t = 1, we have

(5.6)
$$D = 2 \sin \frac{\Psi}{2}$$

and

(5.7)
$$T_S = \frac{r_S}{4\pi} \int \Delta g_M \{ \text{cosec } \frac{\Psi}{2} + 1 - 6 \text{ sin } \frac{\Psi}{2} \}$$

$$-\cos \Psi \left[5 + 3 \ln \left(\sin \frac{\Psi}{2} + \sin^2 \frac{\Psi}{2}\right)\right]\right] d\sigma$$

The quantity

$$(5.8) \quad \zeta = \frac{T}{\gamma}$$

is called the height anomaly and

$$(5.9) \quad \zeta_{S} = \frac{T_{S}}{\gamma_{S}}$$

is approximately the elevation of the geoid above the reference ellipsoid.

Because ζ_S is one of the most important quantities of geodesy, we may often assume that the values of it have already been computed.

Then we can compute the "coating" function of Helmert:

(5.10)
$$\mu = \Delta g + \frac{3}{2} \frac{\zeta_S}{r_S}$$

This function has the advantage that the long formula (5.4) for the computation of T at higher altitudes can be replaced by a shorter one:

(5.11)
$$T_{P} = \frac{r^{2}M}{2\pi r_{P}} \int \frac{\mu}{D} d\sigma$$

In our present problem, we have to compute the components of the gravity disturbance:

$$(5.12) \quad \delta_n = -\frac{\partial T}{\partial r}$$

$$(5.13) \quad \delta_{\rm m} = -\frac{\cos \alpha}{r} \frac{\partial T}{\partial \Psi}$$

$$(5.14) \quad \delta_{\ell} = -\frac{\sin \alpha}{r} \frac{\partial T}{\partial \Psi}$$

If we take T from the long formula (5.4), the result can be written in form

(5.15)
$$\delta_n = \frac{1}{2\pi} \int \Delta g F_1 d\sigma$$

(5.16)
$$\delta_{\rm m} = \frac{1}{2\pi} \int \Delta g F_2 \cos \alpha d\sigma$$

(5.17)
$$\delta_{\ell} = \frac{1}{2\pi} \int \Delta g \, F_{a} \sin \alpha \, d\sigma$$

where

(5.18)
$$F_{1} = \frac{1}{2} t^{2} \left\{ \frac{1-t^{2}}{D^{3}} + \frac{4}{D} + 1 - 6D - t \cos \Psi \left(13 + 6 \ln \frac{D+1-t \cos \Psi}{2} \right) \right\}$$

(5.19)
$$F_2 = t^3 \sin \frac{\pi}{2} \left\{ \frac{1}{D^3} + \frac{3}{D} - 4 + \frac{3}{2} \left(\frac{D-1+t \cos \frac{\pi}{2}}{D \sin^2 \frac{\pi}{2}} - \ln \frac{D+1-t \cos \frac{\pi}{2}}{2} \right) \right\}$$

If we use the short formula (5.11), we obtain

(5.20)
$$\delta_n = \frac{1}{2\pi} \int \mu f_1 d\sigma$$

(5.21)
$$\delta_{\rm m} = \frac{1}{2\pi} \int \mu \, f_2 \cos \alpha \, d\sigma$$

(5.22)
$$\delta_{\ell} = \frac{1}{2\pi} \int \mu f_2 \sin \alpha d\sigma$$

where

(5.23)
$$f_1 = \frac{t^2}{D^3} (1 - t \cos \Psi)$$

$$(5.24)$$
 $f_2 = \frac{t^3}{D^3} \sin \Psi$

Table 5.1 shows some values of F_1 and f_1 , Table 5.2 those of F_2 and f_2 . We see that for small values of Ψ , the functions F and f are almost equal.

In practical computations, we must first find the values of $\,^{\Psi}$ and $\,^{\alpha}$. If the earth is considered as a sphere, we have

(5.25)
$$\cos \Psi = \sin \phi_{P} \sin \phi_{M} + \cos \phi_{P} \cos \phi_{M} \cos (\lambda_{M} - \lambda_{P})$$

(5.26)
$$\sin \Psi \cos \alpha = \cos \phi_P \sin \phi_M - \sin \phi_P \cos \phi_M \cos (\lambda_M - \lambda_P)$$

(5.27).
$$\sin \Psi \sin \alpha = \cos \phi_M \sin (\lambda_M - \lambda_P)$$

For small values of Ψ , we can use the approximate formulas:

$$(5.28) \quad m = \phi_{M} - \phi_{P}$$

Table 5.1

	H = 2 km	H = 8	H = 32	H = 128	H = 512
Ψ		F_1			
0' 1 3 10 30 1° 3 10 30 60 90 120 150 180	1016xxxx 4009xxx 3953xx 1321x 710 182 45 12.1 - 3.7 - 3.3 - 2.3 0.4 3.0 4.1	636xxx 588xxx 3522xx 4002x 2062 356 51 12.2 - 3.7 - 3.3 - 2.3 0.4 3.0 4.1	400xx 398xx 3824x 2594x 5085 947 76 12.8 - 3.6 - 3.3 - 2.3 0.4 3.0 4.0	2559 2558 2552 2481 1980 1120 148 14.9 - 3.4 - 3.1 - 2.2 0.4 2.8 3.8	174 174 174 173 171 162 105 18.8 - 2.7 - 2.4 - 1.8 0.3 2.3 3.1
		f	<u>.</u>		
0' 1 3 10 30 1° 3 10 30 60 90 120 150 180	1016xxxx 4005xxx 3937xx 1267x 528 88 12 2.9 1.0 0.5 0.4 0.3	635xxx 587xxx 3512xx 3956x 1883 262 18 3.1 1.0 0.5 0.4 0.3	397xx 395xx 3795x 2568x 4928 858 44 3.8 1.0 0.5 0.4 0.3 0.3	2480 2480 2472 2403 1908 1061 118 6.3 1.1 0.5 0.3 0.3	155 155 154 152 144 90 12.2 1.3 0.5 0.3 0.3

Table 5.2

ì	H = 2 km	H = 8	H = 32	H = 128	H = 512
Ψ		\mathbb{F}_{2}			
0' 1 3 10 30 1° 3 10 30 60 90 120 150 180	0 3714xxx 1095xxx 1166xx 1327x 3366 396 43.5 7.5 1.2 - 1.4 - 2.2 - 1.5	0 1362xx 2445xx 916xx 1286x 3333 394 43.4 7.5 1.2 - 1.4 - 2.2 - 1.5	0 237x 662x 1494x 859x 2957 385 42.9 7.5 1.2 - 1.3 - 2.1 - 1.5 0	0 36 109 353 842 936 309 40.5 7.1 1.2 - 1.3 - 2.0 - 1.4	0 1 2 6 18 33 62 28.0 5.9 1.1 - 1.0 - 1.7 - 1.2
01	0	f _a		0	0
1 3 10 30 1°	3713xxx 1093xxx 1160xx 1310x 3280	1360xx 2442xx 914xx 1271x 3252	229x 660x 1487x 850x 2894	36 107 348 827 915	1 2 6 16 31
3 10 30 60	364 32•7 3•6 0•9 0•4	363 32.7 3.6 0.9 0.4	357 32.5 3.6 0.9 0.4	289 31.2 3.5 0.8 0.3	72 22.3 3.1 0.8 0.3
120 150 180	0.3 0.3 0.2	0.3 0.3 0.2	0.3 0.3 0.2	0.3 0.3 0.2	0.3 0.2 0.2

(5.29)
$$\ell = (\lambda_{M} - \lambda_{p}) \cos \varphi_{p}$$

(5.30)
$$n = \frac{H}{R} + \frac{1}{2} (m^2 + \ell^2)$$

$$(5.31) \qquad \triangle^2 = n^2 + m^2 + \ell^2$$

$$(5.32)$$
 $f_1 = \frac{n}{\Delta^3}$

$$(5.33) f_2 \cos \alpha = \frac{m}{\Lambda^3}$$

$$(5.34) f2 sin $\alpha = \frac{\ell}{\sqrt{3}}$$$

These formulas can be used only when $\Psi < 20^\circ$. On the other hand, the effect of zones $\Psi > 20^\circ$ is often negligible. Otherwise, this effect may be computed for a few points only and interpolated for the other points. When the components δ_n , δ_m , δ_k have been computed, we must convert them to the system x,y,z by the matrix multiplication:

$$\begin{pmatrix} \delta_{\mathbf{x}} \\ \delta_{\mathbf{y}} \\ \delta_{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} -\cos\varphi\cos\lambda & -\sin\varphi\cos\lambda & -\sin\lambda \\ -\cos\varphi\sin\lambda & -\sin\varphi\sin\lambda & \cos\lambda \\ -\sin\varphi\cos\alpha & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{n}} \\ \delta_{\mathbf{m}} \\ \delta_{\mathbf{y}} \end{pmatrix}$$

The components

 δ_A along the trajectory

 $\boldsymbol{\delta}_{R}^{}$ horizontally to the right hand side

 δ_{D} perpendicularly (not vertically) down

can be obtained as follows. Take three consecutive points P_{i-1} , P_{i} and P_{i+1} of the trajectory. Compute:

(5.36)
$$x' = x_{i+1} - x_{i-1}$$

 $z' = z_{i+1} - z_{i-1}$

$$x'' = \frac{(x'x_1+y'y_1) z_1-z'p_1^2}{r'p_1r_1}$$

(5.37)
$$y'' = \frac{x'y_{i} - y'x_{i}}{r'p_{i}}$$
$$z'' = \frac{x'x_{i} + y'y_{i} + z'z_{i}}{r'p_{i}}$$

$$(5.38)$$
 $p'' = \sqrt{x''^2 + y''^2}$

$$\begin{pmatrix} \delta_{A} \\ \delta_{R} \\ \delta_{D} \end{pmatrix} = \begin{pmatrix} -z'' & -x'' & y'' \\ 0 & \frac{y''}{p''} & \frac{x''}{p''} \\ p'' & -\frac{x''z''}{p''} & \frac{y''z''}{p''} \end{pmatrix} \begin{pmatrix} \delta_{n} \\ \delta_{m} \\ \delta_{\ell} \end{pmatrix}$$

Size of Blocks. Around the point P define four regions, each bounded by latitudes ϕ_1 , ϕ_2 and longitudes λ_1 , λ_2 .

 $1\lambda - \lambda_{p} 1 < 4^{\circ}$

Region
$$C_{3}$$
 $\varphi_{2}-\varphi_{1}=25^{\circ}$ outside B $\lambda_{2}-\lambda_{1}=30^{\circ}$ $|\varphi-\varphi_{P}|<10$ $|\lambda-\lambda_{P}|<12^{\circ}$ 30°

 φ_1 , φ_2 , λ_1 , λ_2 are all divisible by 5°.

Region D outside C

For region A , pick up the mean anomaly cards $5' \times 5'$ ($36 \times 48 = 1728 \text{ cards}$), for region B , $20' \times 20' \text{ cards}$ ($21 \times 27 - 9 \times 12 = 459 \text{ cards}$), for region C , $1^{\circ} \times 1^{\circ}$ cards ($25 \times 30 - 7 \times 9 = 687 \text{ cards}$) and for region D , $5^{\circ} \times 5^{\circ}$ cards. Region D will be used for a few points only.

The influence of zone D on δ_x , δ_y , δ_z can also be computed directly by means of the Cartesian coordinates x,y,z. If these coordinates x,y,z are computed for the centers of the 5°×5° squares, then the integration can be carried out by the formulas

(5.40)
$$\Delta \delta_{x} = \frac{R^{2}}{2\pi} \int_{D} \mu \cdot \frac{x_{M} - x_{P}}{\Delta^{3}} d\sigma$$

$$\Delta \delta_{y} = \frac{R^{2}}{2\pi} \int_{D} \mu \cdot \frac{y_{M} - y_{P}}{\Delta^{3}} d\sigma$$

$$\Delta \delta_{z} = \frac{R^{2}}{2\pi} \int_{D} \mu \cdot \frac{x_{M} - x_{P}}{\Delta^{3}} d\sigma$$

where

(5.41)
$$\triangle^2 = (x_M^- x_p)^2 + (y_M^- y_p)^2 + (z_M^- z_p)^2$$

and $\Delta \delta_x$, $\Delta \delta_y$, $\Delta \delta_z$ are the contribution of the zone D to δ_x , δ_y , δ_z .

bу

H. Moritz

INFLUENCE OF AN INACCURATE GRAVITY MATERIAL

For simplicity, the error propagation will be studied for the coating method.

Since, by eqs. (5.20) - (5.24), the components of the gravity disturbance are given by

$$\delta_{n} = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \cdot \frac{t^{2}}{D^{3}} (1 - t \cos \Psi) \sin \Psi d\Psi d\alpha,$$

$$\begin{cases} \delta_{m} \\ \delta_{\ell} \end{cases} = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \cdot \frac{t^{3}}{D^{3}} \sin^{2} \Psi \begin{cases} \cos \alpha \\ \sin \alpha \end{cases} d\Psi d\alpha,$$

we find for their standard errors m, in, m, t

$$\mathbf{m}_{\mathbf{n}}^{\mathbf{a}} = \frac{1}{4\pi^{\mathbf{a}}} \int_{\mathbf{C}}^{2\pi} \int_{\mathbf{Y}-\mathbf{O}}^{\pi} \int_{\mathbf{Y}-\mathbf{O}}^{2\pi} \int_{\mathbf{Y}-\mathbf{O}}^{\pi} \sigma(\mathbf{Y}, \alpha, \mathbf{Y}', \alpha') \cdot \frac{\mathbf{t}^{4}}{\mathbf{D}^{\mathbf{3}}\mathbf{D}'^{\mathbf{3}}} (1 - \mathbf{t} \cos \mathbf{Y}).$$

·(1 - t cos Ψ ')sin Ψ sin Ψ ' d Ψ d α d Ψ ' d α ',

(6.1)
$$\begin{cases} m^{3} \\ m \\ m^{2} \end{cases} = \frac{1}{4\pi^{3}} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \int_{\alpha'=0}^{\pi} \int_{\psi'=0}^{\pi} \sigma(\Psi,\alpha,\Psi',\alpha') \cdot \frac{t^{6}}{D^{3}D^{13}} \sin^{3}\Psi \sin^{3}\Psi'.$$

$$\left. \begin{array}{l} \cos \alpha \cos \alpha' \\ \sin \alpha \sin \alpha' \end{array} \right\} \mathrm{d} \Psi \; \mathrm{d} \alpha \; \mathrm{d} \Psi' \; \mathrm{d} \alpha',$$

where $\sigma(\Psi,\alpha,\Psi',\alpha')$ is the error covariance function, or error function, of μ [Moritz, 1962a, 1963].

We shall assume uniform coverage of the whole earth by gravity stations so that σ approximately depends on the relative position of points $P(\Psi,\alpha)$ and $P'(\Psi',\alpha')$ only. Furthermore we assume that the error function has a sharp maximum at P'=P and drops off rapidly to zero with increasing distance PP'.

Then we can simplify the integrations considerably by approximately replacing $\sigma(\Psi,\alpha,\Psi^{*},\alpha^{*})$ by

(6.2)
$$\frac{S}{\sin \Psi'} \cdot \delta(\Psi' - \Psi) \delta(\alpha' - \alpha)$$

where $\delta(\Psi' - \Psi)$, $\delta(\alpha' - \alpha)$ are Dirac's delta functions and S is a constant given by

(6.3)
$$S = \int_{\alpha'=0}^{2\pi} \int_{\Psi'=0}^{\pi} \sigma(\Psi,\alpha,\Psi',\alpha') \sin \Psi' d\Psi' d\alpha'.$$

Since according to a property of the delta function

$$\int f(x') \delta(x' - x) dx' = f(x),$$

the integrations with respect to $\Psi^{,}$ $\alpha^{,}$ can be performed immediately and we get

$$m_{n}^{2} = \frac{S}{4\pi^{2}} \int_{\Omega-\Omega}^{2\pi} \int_{\Psi-\Omega}^{\pi} \frac{t^{4}}{D^{6}} (1 - t \cos \Psi)^{2} \sin \Psi d\Psi d\alpha,$$

$$\begin{Bmatrix} m^{2} \\ m \\ m^{2} \\ \ell \end{Bmatrix} = \frac{S}{4\pi^{2}} \int_{\Omega = 0}^{2\pi} \int_{\Psi = 0}^{\pi} \frac{t^{6}}{D^{6}} \sin^{3}\Psi \begin{Bmatrix} \cos^{2}\alpha \\ \sin^{2}\alpha \end{Bmatrix} d\Psi d\alpha,$$

and finally,

(6.4)
$$m_{n}^{3} = \frac{S}{2\pi} \left[\frac{t^{4}(t^{3}+1)}{2(t^{3}-1)^{3}} + \frac{t^{4}}{t^{3}-1} - \frac{t^{3}}{4} \ln \left| \frac{t-1}{t+1} \right| \right],$$

$$m_{m}^{3} = m_{\ell}^{3} = \frac{S}{2\pi} \left[\frac{t^{4}(t^{3}+1)}{4(t^{3}-1)^{3}} + \frac{t^{3}}{8} \ln \left| \frac{t-1}{t+1} \right| \right].$$

Since

$$t = \frac{R}{R + H} = 1 - \frac{H}{R} + \cdots$$

(R:mean radius of the earth) we can develop in series, and neglecting higher powers of $\frac{H}{R}$ we get

(6.5)
$$m_n^2 = \frac{SR^2}{8\pi H^2}$$
, $m_m^2 = m_\ell^2 = \frac{SR^2}{16\pi H^2}$.

(For H = 500 km the error of these approximations is smaller than 7%.)

Validity of the Simplified Integration

Now we have to investigate in more detail the validity of replacing the error function σ by a product of Delta functions. For simplicity we limit ourselves to the plane approximation. In this case, eqs. (5.20) - (5.22) are simplified to

$$\delta_{n} = \frac{H}{2\pi} \int_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^{2}+x^{2}+y^{2})^{3/2}} dx dy ,$$

$$\begin{cases} \delta_{m} \\ \delta_{y} \end{cases} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^{2}+x^{2}+y^{2})^{3/2}} \begin{Bmatrix} x \\ y \end{Bmatrix} dx dy ,$$

where the xy-plane is horizontal and the z-axis contains point P.

These equations have the general form

(6.7)
$$\iint_{-\infty}^{\infty} \mu(x,y) f(x,y) dx dy$$

and the mean square error is given by

(6.8)
$$m^{2} = \iiint_{-\infty}^{\infty} \sigma(x,y,x',y') f(x,y) f(x',y') dx dy dx' dy'.$$

As before we assume the error function $\sigma(x,y,x',y')$ to have appreciable magnitude only for (x',y') = (x,y) and to be almost zero elsewhere. Then, only the points (x',y') = (x,y) will contribute significantly to the integral and if we put

$$x' = x + \xi$$
, $y' = y + \eta$,

then only small values ξ , η need to be considered. If the usual conditions of continuity and differentiability are satisfied, $f(x',y') \text{ can be developed in a Taylor series with respect to } \xi, \, \eta\text{:}$

$$f(x',y') = f(x,y) + \xi f_{x} + \eta f_{y} + \frac{1}{2} \xi^{3} f_{xx} + \xi \eta f_{xy} + \frac{1}{2} \eta^{3} f_{yy} + \dots$$

where f , f , ... denote partial derivatives at the point (x,y) and $\xi = x' - x \ , \quad \tilde{\eta} = y' - y \ .$

Inserting this in (6.8) we can separate the integrations over x, y and over x, y, getting

$$m^{2} = \iint_{-\infty}^{\infty} \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint_{-\infty}^{\infty} [f(x,y)]^{2} \, dx \, dy + \\
+ \iint_{-\infty}^{\infty} (x'-x) \, \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint_{-\infty}^{\infty} ff_{x} \, dx \, dy + \\
+ \iint_{-\infty}^{\infty} (y'-y) \, \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint_{-\infty}^{\infty} ff_{y} \, dx \, dy + \\
+ \frac{1}{2} \iint_{-\infty}^{\infty} (x'-x)^{2} \, \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint_{-\infty}^{\infty} ff_{xx} \, dx \, dy + \\
+ \iint_{-\infty}^{\infty} (x'-x)(y'-y) \, \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint_{-\infty}^{\infty} ff_{xy} \, dx \, dy + \\
+ \frac{1}{2} \iint_{-\infty}^{\infty} (y'-y)^{2} \, \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint_{-\infty}^{\infty} ff_{yy} \, dx \, dy \cdot \dots ,$$

provided $\sigma(x,y,x',y')$ depends on x-x', y-y' only, i.e., we have the same accuracy everywhere. Furthermore, if σ is a symmetrical function of x'-x and y'-y which is a very natural assumption, then the integrals containing x'-x and y'-y linearly will vanish and there remains

(6.9)
$$m^{2} = S_{0} \iint_{-\infty}^{\infty} [f(x,y)]^{2} dx dy + \frac{1}{2} S_{1} \iint_{-\infty}^{\infty} ff_{xx} dx dy + \frac{1}{2} S_{2} \iint_{-\infty}^{\infty} ff_{yy} dx dy ...$$

where

$$S_{o} = \iint_{-\infty}^{\infty} \sigma \, dx^{i} \, dy^{i}, \quad S_{1} = \iint_{-\infty}^{\infty} (x^{i}-x)^{2} \, \sigma dx^{i} \, dy^{i},$$

$$S_{2} = \iint_{-\infty}^{\infty} (y^{i}-y)^{2} \sigma \, dx^{i} \, dy^{i}.$$

If we would have replaced, as before, the error function σ by a product of delta functions,

$$S_0 \cdot \delta(x!-x) \delta(y!-y)$$
,

we would have got

$$m^2 = S_0 \int_{-\infty}^{\infty} [f(x,y)]^2 dx dy$$

which is the first and principal term of (6.9).

Thus the use of delta functions is justified. Transferring these results from the plane to the spherical case offers no difficulties. S_o , the integral of the error function over the plane (6.10), is replaced by S, the integral over the unit sphere (6.3); we have the relation

(6.11)
$$S = \frac{S_0}{R^2}$$

where the scaling factor R² represents the transition from the terrestrial sphere (or its tangential plane) to the unit sphere.

The error of the delta function method is the effect of S_1 and S_2 in (6.9).

In order to evaluate the quantities S, S_0 , S_1 , S_2 we have to make some assumption concerning the error function σ . If this function has the character assumed above - everywhere the same form, sharp maximum at x' = x, y' = y, vanishing farther away from this

point - then, within the limit of the obtainable accuracy, we can usually take the well-known function

(6.12)
$$\sigma(x,y,x',y') = \sigma_0 e^{-c^2[(x'-x)^2 + (y'-y)^2]}$$

(6.10) can then be evaluated immediately to give

(6.13)
$$S_0 = \frac{\pi \sigma_0}{c^2}$$
, $S_1 = S_2 = \frac{\pi \sigma_0}{2c^4}$

and

(6.14)
$$S = \frac{\pi \sigma_0}{e^2 R^2}$$
.

Now we can return to eqs. (6.6). First we consider δ_n . Here

$$f(x,y) = \frac{H}{2\pi} \frac{1}{(H^2 + x^2 + y^3)^{3/2}}$$
.

Differentiating twice, inserting in (6.9) and integrating yields

$$m_n^2 = \frac{S_0}{8\pi H^2} - \frac{3(S_1 + S_2)}{64\pi H^4}$$
.

The first term is clearly the same as the first equation in (6.5).

Inserting (6.13) we finally get

(6.15)
$$m_n^2 = \frac{\sigma_0}{8c^2H^2} (1 - \frac{3}{8c^2H^2} + \dots)$$

and in a similar way we find

(6.16)
$$m_{\rm m}^2 = m_{\ell}^2 = \frac{\sigma_{\rm o}}{16c^2H^2} \left(1 - \frac{3}{8c^2H^2} + \dots \right)$$
.

The correction term in the parenthesis is practically zero for elevations H >> c⁻¹ , but in view of the practical purpose of these

accuracy formulas, it can already be neglected for $H = c^{-1}$.

Discussion

To sum up the formulas we have for the standard errors of $\delta_n,$ $\delta_m,$ δ_l :

(6.17)
$$m_n = \frac{k_1}{H}, m_m = m_\ell = \frac{k_2}{H}$$

where

(6.18)
$$k_1 = \sqrt{\frac{SR^3}{8\pi}}, k_2 = \sqrt{\frac{SR^3}{16\pi}}$$

and

$$S = \frac{\pi_{\sigma_o}}{R^2 c^2}.$$

These formulas are valid for elevations $H > c^{-1}$. For smaller elevations they cannot be applied (for H = 0 they would yield ∞ , which is obviously wrong).

The error covariance function σ is supposed to have the form (6.12):

(6.19)
$$\sigma(x,y,x',y') = \sigma_0 e^{-c^2 s^2}$$

where s is the distance of the points (x,y) and (x',y'). According to the derivation of the above formulas, σ is the error function of μ , the density of the coating. Since $\mu = \Delta g + \frac{3\gamma}{2R} N$, it contains the influence of errors in Δg and in N . The errors of N have a

small influence but they diminish much less with increasing distance and can hardly be represented in the form (6.19). So, in order to apply our formulas, we must be able to neglect the errors of N. That this is possible can be seen in the following way.

We would have avoided the trouble concerning the errors of N, had we used the direct formulas (5.15) - (5.19) instead of using the coating method (5.20) - (5.24). These formulas are, however, very difficult to handle for our purpose. On the other hand, for smaller values of ψ , where the effect of errors in μ is largest, F and f are closely equal so that, for this particular purpose, we might replace μ in (5.20) - (5.24) by Δg . Then we can take σ in (6.1) and in the subsequent developments to be the error covariance function of the gravity anomalies Δg only. From this we can conclude that the error of (6.17) due to neglecting the inaccuracy of N must be small.

Thus, we can consider σ in (6.19) to be the error function for the interpolation of gravity anomalies. Then,

$$\sigma_0 = m^2$$

is the square of the average standard error of interpolation, m . In Table 6.1 some numerical values of m and c are given, together with the constants k_1 and k_2 defined above.

Table 6.1 Constants k_1 , k_2 for average station distances s

s km	m mgal	e -1 km	k ₁ mg a l.km	k ₂ mgal.km
10	1	13	2.5	1.8
50	9	25	45	32
100	14	45	126	89
200	16	90	287	203

To get the standard errors of δ and of δ , δ , the above values for k_1 and k_2 have to be divided by the elevation H in kilometers.

For the computation of the dependence of m and c on the average station distances, an idealized gravity station net consisting of equilateral triangles with side s has been assumed. The computation was done in a way described in [Moritz, 1962b, Appendix], using numerical values for the covariance function of gravity anomalies given in [Kaula, 1959] combined with the standard interpolation errors of [Molodenskii et. al., 1962, p. 172]. Since the method used is not quite rigorous, the results of Table 2 must be considered somewhat preliminary.

Error Correlation of δ_n , δ_m , δ_ℓ

For the error covariances of δ_n , δ_m , δ_ℓ in two points. P, P' we have found [Moritz, 1963]:

$$\sigma_{n,PP'} = \frac{\sigma_0}{8c^{3}H^{3}} \left[1 - \frac{3}{8} \frac{s^{3}}{H^{3}} + \dots \right],$$

(6.20)
$$\sigma_{\text{m,PP}} = \frac{\sigma_{\text{o}}}{16c^{2}H^{2}} \left[1 - \frac{3}{16} \frac{s^{2}}{H^{2}} (1 + 2 \cos^{2}\alpha) + \dots \right],$$

$$\sigma_{\ell,PP'} = \frac{\sigma_0}{16c^2H^2} \left[1 - \frac{3}{16} \frac{s^2}{H^2} (1 + 2 \sin^2 \alpha) + \dots \right].$$

P, P' are assumed to have the same elevation H, s is the distance and α is the azimuth from P to P'. These formulas, being the first terms of power series, are valid for small distances s only. For P = P', by (6.5) and (6.14)

(6.21)
$$\sigma_{n,PP} = m_{n}^{2}, \sigma_{m,PP} = m_{m}^{2}, \sigma_{\ell,PP} = m_{\ell}^{2};$$

the error covariances become variances, i.e., the squares of the standard errors of δ_n , δ_m , δ_ℓ .

Dividing the covariances (6.20) by the variances (6.21), we get the correlation coefficients

$$\rho_{n,PP}^{} = 1 - \frac{3}{8} \frac{s^{2}}{H^{2}} + \cdots,$$

$$(6.22) \quad \rho_{m,PP}^{} = 1 - \frac{3}{16} \frac{s^{2}}{H^{2}} (1 + 2 \cos^{2}\alpha) + \cdots,$$

$$\rho_{\ell,PP}^{} = 1 - \frac{3}{16} \frac{s^{2}}{H^{2}} (1 + 2 \sin^{2}\alpha) + \cdots.$$

 $ho = \pm$ 1 means maximum correlation, i.e., complete functional dependence; ho = 0 means independence (more strictly speaking, complete lack of correlation). For a definite s, the correlation coefficients (6.22) will be the closer to 1, the greater the altitude H is. Thus, with increasing altitude, the standard errors of δ_n , δ_m , δ_ℓ decrease, but the correlation increases. Since the accuracy is characterized by the standard errors and the error correlation, this is significant, too.

7. INFLUENCE OF THE REMOTE ZONES

The purpose of the estimation of the influence of the distant zones is to decide how far the integration must be extended if certain accuracy requirements are prescribed.

Since the effect of the distant zones is almost independent of the elevation H , we may, for simplicity, put H=0, i.e., we consider P to be situated directly on the geoid. Then, since

$$\delta_{n} = \Delta g + \frac{2\gamma}{R} N, \quad \delta_{m} = \gamma \xi, \quad \delta_{\ell} = \gamma \eta,$$

we can, for the direct method, use one of the evaluations of the effect of distant zones on the components ξ , η of the deflection of the vertical and on N (e.g., [Kaula, 1957]). For our purpose it is best to use the formulas of [Molodenskii et al., 1962] which can also be

easily adapted to the coating method.

Direct Method (using Δg). The disturbing potential is

(7.1.a)
$$T = \gamma N = \frac{R}{4\pi} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \Delta g S(\cos \Psi) \sin \Psi d\Psi d\alpha$$

where $S(\cos \Psi)$ is Stokes' function

(7.2.a)
$$S(\cos \Psi) = \frac{1}{\sin \frac{\Psi}{2}} - 3 \cos \Psi \ln \left(\sin \frac{\Psi}{2} + \sin^2 \frac{\Psi}{2}\right) - 6 \sin \frac{\Psi}{2} + 1 - 5 \cos \Psi$$
.

If we extend the integration with respect to Ψ only up to a spherical radius $\Psi < \pi$, we commit an error.

(7.3.a)
$$\Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=\Psi}^{\pi} \Delta g S(\cos \Psi) \sin \Psi d\Psi d\alpha$$

which can be also written

(7.4.a)
$$\Delta N = \frac{R}{4\pi \psi} \int_{\alpha=0}^{2\pi} \int_{\gamma=0}^{\pi} \Delta g \, \overline{S}(\cos \gamma) \sin \gamma \, d\gamma \, d\alpha$$

where

(7.5.a)
$$\overline{S}(\cos \Psi) = \begin{cases} 0 & \text{if } \Psi < \Psi \\ 0 & \text{o} \end{cases}$$
.

Expand S(cos Y) in a series of Legendre's polynomials:

(7.6.a)
$$\overline{S}(\cos \Psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n P_n (\cos \Psi)$$

where the coefficients Q_n depend on Ψ . Then, by multiplying both sides by $P_n(\cos \Psi)$ sin Ψ and integrating from 0 to π we find

$$Q_{n} = \int_{0}^{\pi} \frac{1}{S(\cos \Psi)} P_{n}(\cos \Psi) \sin \Psi d\Psi = \int_{\Psi}^{\pi} S(\cos \Psi) P_{n}(\cos \Psi) \sin \Psi d\Psi.$$

By substituting

(7.7.a)
$$z = \sin \frac{\Psi}{2}$$
, $t = \sin \frac{\Psi_0}{2}$

we get

(7.8.a)
$$Q_n = -4 \int_1^t P_n (1 - 2z^2) S(1 - 2z^2) z dz$$
.

Performing the integration yields (we need the Q_n only for $n \ge 2$):

$$Q_2 = 2 - 4t + 5t^2 + 14t^3 - \frac{53}{2}t^4 - 30t^5 + 47t^6 + 18t^7 - \frac{51}{2}t^8 + (6t^2 - 24t^4 + 36t^6 - 18t^8) \ln t(1+t),$$

(7.9.a)
$$Q_3 = 1 - 4t + 5t^2 + 22t^3 - 46t^4 - \frac{372}{5}t^5 + 136t^6 + 104t^7 - 166t^8 - 48t^9 + \frac{352}{5}t^{10} + (6t^2 - 42t^4 + 108t^6 - 120t^8 + 48t^{10}) \ln t(1+t),$$

$$Q_{4} = \frac{2}{3} - 4t + 5t^{2} + \frac{98}{3}t^{3} - 72t^{4} - 156t^{5} + 320t^{6} + 360t^{7} - 645t^{8}$$
$$- \frac{1120}{3}t^{9} + 602t^{10} + 140t^{11} - 210t^{12} + (6t^{2} - 66t^{4} + 260t^{6} - 480t^{8} + 420t^{10} - 140t^{12}) \ln t(1+t).$$

Higher order Q (up to 8th order) may be found in [Molodenskii et al., 1962, p. 149].

By (7.4.a) and (7.6.a) we get

(7.10.a)
$$\Delta N = \frac{R}{2\gamma_{n=2}}^{\infty} Q_n \Delta g_n$$

if

$$\Delta g = \sum_{n=2}^{n} \Delta g_n$$

is the development of Δg in Laplace's spherical harmonics ($\Delta g_{_{\rm O}}$ and Δg_{1} are missing, as usual).

The coating method can be treated in an exactly analogous way. Coating Method (using μ). The disturbing potential is

(7.1.b)
$$T = \gamma N = \frac{R}{4\pi} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \mu M (\cos \Psi) \sin \Psi d\Psi d\alpha$$

where

$$(7.2.b) \quad M(\cos \Psi) = \frac{1}{\sin \frac{\Psi}{2}}.$$

If we extend the integration with respect to $\,\Psi\,$ only up to a spherical radius $\,\Psi\,$ < π , we commit an error

(7.3.b)
$$\Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=\Psi}^{\pi} \mu M (\cos \Psi) \sin \Psi d\Psi d\alpha$$

which can be also written

$$(7.4.b) \qquad \Delta N = \frac{R}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\Psi=0}^{\pi} \mu \, \overline{M} \, (\cos \Psi) \sin \Psi \, d\Psi \, d\alpha$$

where

$$(7.5.b) \quad \overline{M}(\cos \Psi) = \begin{cases} 0 & \text{if } \Psi < \Psi \\ M (\cos \Psi) = \frac{1}{\sin \frac{\Psi}{2}} & \text{if } \Psi \ge \Psi \end{cases}.$$

Expand $M(\cos \Psi)$ in a series of Legendre's polynomials:

(7.6.b)
$$\frac{-}{M}(\cos \Psi) = \sum_{n=0}^{\infty} (n-1) q_n P_n (\cos \Psi)$$

where the coefficients q depend on Ψ . Then, by multiplying both sides by $P_n(\cos\Psi)$ sin Ψ and integrating from 0 to π we find

$$q_{n} = \frac{2n+1}{2(n-1)} \int_{0}^{\pi} \overline{M} (\cos \Psi) P_{n} (\cos \Psi) \sin \Psi d\Psi =$$

$$= \frac{2n+1}{2(n-1)} \int_{\Psi}^{\pi} \frac{1}{\sin \Psi} P_{n} (\cos \Psi) \sin \Psi d\Psi .$$

By substituting

$$(7.7.a) z = \sin \frac{\Psi}{2}, t = \sin \frac{\Psi_0}{2}$$

we get

(7.8.b)
$$q_n = \frac{2(2n+1)}{n-1} \int_{\pm}^{1} P_n(1-2z^2) dz$$
.

Performing the integration yields (we need the q_n only for $n \ge 2$):

$$q_{2} = \frac{2}{1}(1 - 5t + 10t^{3} - 6t^{5}),$$

$$q_{3} = \frac{2}{2}(1 - 7t + 28t^{3} - 42t^{5} + 20t^{7}),$$

$$q_{4} = \frac{2}{3}(1 - 9t + 60t^{3} - 162t^{5} + 180t^{7} - 70t^{9}),$$

$$q_{5} = \frac{2}{4}(1 - 11t + 110t^{3} - 462t^{5} + 880t^{7} - 770t^{9} + 252t^{11}),$$

$$(7.9.b) \quad q_{6} = \frac{2}{5}(1 - 13t + 182t^{3} - 1092t^{5} + 3120t^{7} - 4550t^{9} + 3276t^{11} - 924t^{13}),$$

$$q_{7} = \frac{2}{6}(1 - 15t + 280t^{3} - 2268t^{5} + 9000t^{7} - 19250t^{9} + 22680t^{11} - 13860t^{13} + 3432t^{15}),$$

$$q_{8} = \frac{2}{7}(1 - 17t + 408t^{3} - 4284t^{5} + 22440t^{7} - 65450t^{9} + 111384t^{11} - 109956t^{13} + 58344t^{15} - 12870t^{17}).$$

By (7.4.b) and (7.6.b) we get

$$\Delta N = \frac{R}{\gamma} \sum_{n=2}^{\infty} \frac{n-1}{2n+1} q_n \mu_n$$

if

$$\mu = \sum_{n=2}^{n} \mu_n$$

if the development of $\,\mu\,$ in Laplace's spherical harmonics ($\mu_{_{\rm O}}$ and μ_{1} are missing together with $\Delta g_{_{\rm O}}$ and $\Delta g_{1}).$

If

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n$$

then

$$N = \frac{R}{\gamma} \sum_{n=2}^{\infty} \frac{\Delta g_n}{n-1} .$$

Therefore,

$$\mu = \Delta g + \frac{3\gamma}{2R} N = \sum_{n=2}^{\infty} \frac{2n+1}{2(n-1)} \Delta g_n$$

so that

$$\mu_{n} = \frac{2n+1}{2(n-1)} \Delta g_{n} .$$

Inserting this in the formula for N we finally obtain

(7.10.b)
$$\Delta N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} q_n \Delta g_n$$

which is analogous to (7.10.a).

Mean Square Effect on N, ξ , η . Squaring (7.10.a) and averaging over the whole earth we get

$$\overline{\Delta N}^{2} = \frac{R^{2}}{4\gamma^{2}} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_{n'} \cdot \overline{\Delta g_{n} \Delta g_{n'}}.$$

Since the integral, over the sphere, of the product of two Laplace harmonics of different degree is zero (Orthogonality !), only the terms where n' = n remain and we obtain

(7.11.a)
$$\overline{\Delta N}^2 = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} Q_n^2 \overline{\Delta g}_n^2$$

where $\overline{\Delta N}$ is the root mean square influence on N of the zones beyond a spherical radius Ψ_o , and $\overline{\Delta g}_n$ is the root mean square average of the Laplace spherical harmonic Δg_n .

The derivation of the RMS influence on the components ξ , η of the deflection of the vertical is rather involved. Since it is omitted in [Molodenskii et. al., 1962], it may be of interest to give it here.

By (7.10.a),

$$\Delta \xi = \frac{1}{R} \frac{\partial N}{\partial \phi} = \frac{1}{2\gamma} \sum_{n=2}^{\infty} Q_n \frac{\partial \Delta g_n}{\partial \phi} ,$$

$$(7.12)$$

$$\Delta \eta = \frac{1}{R \cos \phi} \frac{\partial N}{\partial \lambda} = \frac{1}{2\gamma \cos \phi} \sum_{n=2}^{\infty} Q_n \frac{\partial \Delta g_n}{\partial \lambda}$$

for the direct method. (For the coating method replace Q by q_n .) Hence,

$$\begin{split} \overline{\Delta \xi^2} &= \frac{1}{4 \gamma^2} \sum_{n=2}^{\infty} \sum_{\substack{n'=2 \\ n = 2}}^{\infty} Q_n Q_n, \quad \overline{\frac{\partial g_n}{\partial \phi} \frac{\partial g_n!}{\partial \phi}} , \\ \overline{\Delta \eta^2} &= \frac{1}{4 \gamma^2} \sum_{n=2}^{\infty} \sum_{\substack{n'=2 \\ n = 2}}^{\infty} Q_n Q_n, \quad \overline{\frac{1}{\cos^2 \phi} \frac{\partial g_n}{\partial \lambda} \frac{\partial g_n!}{\partial \lambda}} , \end{split}$$

so that

$$(7.13) \quad \overline{\Delta \xi^2} + \overline{\Delta \eta^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_{n'} Q_{n'} \cdot \frac{\partial g_n}{\partial \varphi} \frac{\partial g_{n'}}{\partial \varphi} + \frac{1}{\cos^2 \varphi} \frac{\partial g_n}{\partial \lambda} \frac{\partial g_{n'}}{\partial \lambda} \cdot \frac{$$

(The bar denotes averaging over the sphere.)

(7.14)
$$\Delta g_n = \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_n^m (\sin \phi)$$

where a , b are constant coefficients and ϕ , λ are geographical coordinates. Now we can differentiate with respect to ϕ and λ :

$$\frac{\partial \Delta g_{n}}{\partial \varphi} = \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) \frac{dP_{n}}{d\varphi},$$

$$\frac{\partial \Delta g_{n}}{\partial \lambda} = \sum_{m=0}^{n} (mb_{nm} \cos m\lambda - ma_{nm} \sin m\lambda) P_{n}^{m}.$$

Hence we get, taking $n' \ge n$,

$$4\pi \frac{\partial \Delta g_{n}}{\partial \phi} \frac{\partial \Delta g_{n}}{\partial \phi} = \int_{\lambda=0}^{2\pi} \int_{\phi=-\pi/2}^{\pi/2} \frac{\partial \Delta g_{n}}{\partial \phi} \frac{\partial \Delta g_{n}}{\partial \phi} \cos \phi \, d\phi \, d\lambda =$$

(7.16)
$$= 2\pi \operatorname{a}_{\text{no n'o}} \int_{-\pi/2}^{\pi/2} \frac{dP \circ dP \circ}{d\varphi} \frac{dP \circ dP \circ}{d\varphi} \cos \varphi \, d\varphi +$$

$$+ \pi \sum_{m=1}^{n} \left(a_{nm} a_{nm} + b_{nm} b_{nm} \right) \int_{-\pi/2}^{\pi/2} \frac{dP_{n}}{d\phi} \frac{dP_{n}}{d\phi} \cos \phi d\phi.$$

Here we have used the well-known orthogonality relations

$$\int\limits_{0}^{2\pi}\cos\,p\lambda\,\cos\,q\lambda\,\,d\lambda\,=\left\{ \begin{array}{ll} 0,\;p\neq q\\ \pi,\;p=q\neq 0\\ 2\pi,\;p=q=0 \end{array} \right. ,$$

(7.17)
$$\int_{0}^{2\pi} \cos p\lambda \sin q\lambda \, d\lambda = 0 ,$$

$$\int_{0}^{2\pi} \sin p\lambda \sin q\lambda \, d\lambda = \begin{cases} 0, p \neq q \\ \pi, p = q \neq 0 \\ 0, p = q = 0 \end{cases}.$$

Similarly

$$4\pi \frac{\frac{1}{\cos^{2}\varphi} \frac{\partial \Delta g_{n}}{\partial \lambda} \frac{\partial \Delta g_{n'}}{\partial \lambda} = \int_{\lambda=0}^{2\pi} \int_{\varphi=-\pi/2}^{\pi/2} \frac{1}{\cos^{2}\varphi} \frac{\partial \Delta g_{n}}{\partial \lambda} \frac{\partial \Delta g_{n'}}{\partial \lambda} \cos\varphi \, d\varphi \, d\lambda -$$

$$(7.18)$$

$$= \pi \sum_{m=1}^{n} m^{2} (a_{nm}^{2} a_{n'm} + b_{nm}^{2} b_{n'm}) \int_{-\pi/2}^{\pi/2} \frac{P_{n}^{m} P_{n'}^{m}}{\cos^{2}\varphi} \cos\varphi \, d\varphi .$$

Putting

$$\sin \varphi = x$$
, $\cos \varphi d\varphi = dx$

we find by formulas of [Molodenskii et.al., 1962, pp. 165-6]

$$\int_{-\pi/2}^{\pi/2} \frac{dP_n^m}{d\varphi} \frac{dP_n^m}{d\varphi} \cos \varphi \, d\varphi = \int_{-1}^{1} \frac{dP_n^m}{d\varphi} \frac{dP_n^m}{d\varphi} \, dx =$$

$$= \begin{cases} \frac{2n(n+1)}{2n+1} \cdot \frac{(n+m)!}{(n-m)!} - m^2 C_n^m, & \text{if } n' = n, \\ -m^2 C_n^m, & \text{if } n' - n \text{ is even, } n' > n, \\ 0, & \text{if } n' - n \text{ is odd;} \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} \frac{P_n P_n^m}{\cos^2 \varphi} \cos \varphi \, d\varphi = \int_{-1}^{1} \frac{P_n P_n^m}{1-x^2} \, dx =$$

$$= \begin{cases} C_{n}^{m}, & \text{if } n' = n, \\ C_{n}^{m}, & \text{if } n' - n \text{ is even, } n' > n, \\ 0, & \text{if } n' - n \text{ is odd,} \end{cases}$$

where

$$C_n^m = 2(2n - 1) \frac{(n+m-2)!}{(n-m)!} + 2(2n-5) \frac{(n+m-4)!}{(n-m-2)!} + \dots$$

(Concerning C_n^m , there is a misprint on p. 165, op. cit.: " + C_n^m " should read " = C_n^m ".)

Hence we find, for $n' \neq n$:

$$\mu_{\pi} \frac{\partial \phi}{\partial \Delta_{g}} \frac{\partial \phi}{\partial \phi} = \begin{cases} \pi \sum_{m=1}^{n} (a_{nm} a_{nm} + b_{nm} b_{nm}) \cdot (-m^{2} C_{n}^{m}), \\ m=1 \end{cases}$$

$$n! - n \text{ even},$$

$$0, n! - n \text{ odd},$$

$$4\pi \frac{1}{\cos^2 \varphi} \frac{\partial \Delta g_n}{\partial \lambda} \frac{\partial \Delta g_n}{\partial \lambda} = \begin{cases} \pi \sum_{m=1}^{n} (a_{nm} a_{n'm} + b_{nm} b_{n'm}) \cdot m^2 C_n^m, \\ m=1 \end{cases}$$

$$0, n' - n \text{ odd}$$

so that

(7.19)
$$\frac{\partial \Delta g_n}{\partial \varphi} \frac{\partial \Delta g_n}{\partial \varphi} + \frac{1}{\cos^2 \varphi} \frac{\partial \Delta g_n}{\partial \lambda} \frac{\partial \Delta g_n}{\partial \lambda} = 0$$

for $n' \neq n$.

For $n^{\dagger} = n$ we find in the same way

$$\left(\frac{\partial \Delta g_n}{\partial \phi}\right)^2 + \frac{1}{\cos^2 \phi} \left(\frac{\partial \Delta g_n}{\partial \lambda}\right)^2 = \frac{n(n+1)}{2n+1} a_{no}^2 + \frac{1}{2} \frac{n(n+1)}{2n+1} \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!} (a_{nm}^2 + b_{nm}^2).$$

Forming the average of Δg_n^2 itself we find easily

$$\frac{\Delta g_n}{2} = \frac{1}{\mu_n} \int_{\lambda=0}^{2\pi} \int_{\phi=-\pi/2}^{\pi/2} \Delta g_n \cos \phi \, d\phi \, d\lambda =$$
(7.20)

$$= \frac{1}{2n+1} a_{no}^{2} + \frac{1}{2(2n+1)} \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!} (a_{nm}^{2} + b_{nm}^{2})$$

so that

(7.21)
$$\overline{\left(\frac{\partial \Delta g_n}{\partial \varphi}\right)^2 + \frac{1}{\cos^2 \varphi} \left(\frac{\partial \Delta g_n}{\partial \lambda}\right)^2} = n(n+1) \overline{\Delta g_n}^2 .$$

Taking (7.19) and (7.21) into account, we find by (7.13)

$$\overline{\Delta \xi^2} + \overline{\Delta \eta^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} n(n+1) Q_n^2 \overline{\Delta g_n^2}.$$

For reasons of symmetry, $\overline{\Delta \xi}$ and $\overline{\Delta \eta}$ can be taken to be equal, and we finally get

(7.22.a)
$$\overline{\Delta \xi^{a}} = \overline{\Delta \eta^{a}} = \frac{1}{8 \gamma^{a}} \sum_{n=2}^{\infty} n(n+1) Q_{n}^{a} \overline{\Delta g_{n}^{a}}$$

for the direct method.

For the $\underline{\text{coating method}}$ we must replace Q_n by q_n , obtaining

(7.11.b)
$$\overline{\Delta N}^{3} = \frac{R^{2}}{4\sqrt{3}} \sum_{n=2}^{\infty} q_{n}^{3} \overline{\Delta g_{n}}^{3}$$
,

(7.22.b)
$$\overline{\Delta \xi}^{s} = \overline{\Delta \eta}^{2} = \frac{1}{8\gamma^{2}} \sum_{n=2}^{\infty} n(n+1) q_{n}^{2} \overline{\Delta g_{n}^{2}}$$
.

Relationship between Δg_n and the Covariance Function. Comparing eq. (7.20) with eq. (33) of [Kaula, 1959] and taking the relationship between conventional and fully normalized spherical harmonics into account, we find that

$$(7.23) \qquad \overline{\Delta g_n}^a = c_n,$$

the coefficient of n-th degree in the development of the covariance function of gravity anomalies, C(s), in a series of Legendre's polynomials:

(7.24)
$$C(s) = \sum_{n=2}^{\infty} c_n P_n(\cos s).$$

Mean Square Effect on δ_N , δ_M , δ_L . In order to avoid confusion, we shall change our notation of the components of the gravity disturbance vector, now writing as subscripts, capital letters N, M, L instead of n, m, ℓ .

Then,

(7.25a)
$$\delta_{N} = \Delta g + \frac{2\gamma}{R} N$$
, $\delta_{M} = \gamma \xi$, $\delta_{L} = \gamma \eta$.

Denoting the RMS effect of the zones beyond Ψ on these components by $\overline{\Delta\delta}_N$, $\overline{\Delta\delta}_M$, $\overline{\Delta\delta}_L$, respectively, we find from (7.11.a,b) and (7.22.a,b), taking (7.23) into account, the following formulas:

Direct Method

$$\overline{\Delta\delta_{N}} = \sum_{n=2}^{\infty} Q_{n}^{2} c_{n},$$

$$(7.26.a)$$

$$\overline{\Delta\delta_{M}}^{2} = \overline{\Delta\delta_{L}}^{2} = \frac{1}{8} \sum_{n=2}^{\infty} n(n+1) Q_{n}^{2} c_{n}.$$

If we use the coating μ , then

(7.25.b)
$$\delta_{N} = \mu + \frac{\gamma}{2R}N$$
, $\delta_{M} = \gamma \xi$, $\delta_{L} = \gamma \eta$,

and we have instead:

Coating Method

$$\overline{\Delta\delta_{N}}^{\mathbf{a}} = \frac{1}{16} \sum_{n=2}^{\infty} q_{n}^{\mathbf{a}} c_{n},$$

$$(7.26.b) \overline{\Delta\delta_{M}}^{\mathbf{a}} = \overline{\Delta\delta_{L}}^{\mathbf{a}} = \frac{1}{8} \sum_{n=2}^{\infty} n(n+1) q_{n}^{\mathbf{a}} c_{n}.$$

Numerical Results. First we give tables for the Q (Table 7.1.a, taken from [Molodenskii et.al., 1962]), and for q (Table 7.1.b, computed by eqs. (7.9.b)).

Table 7.1.a

Coefficients Q for the direct method as functions of radius Ψ ¥ 0 **Q**6 Q_3 Q_{j_L} Q_{5} **Q**8 Q.7 00 +0.667 +2.000 +1.000 +0.500 +0.400 +0.333 +0.286 5° +1.801 +0.802 +0.470 +0.304 +0.206 +0.141 8.99°(1000km) +1.634 +0.639 +0.312 +0.152 +0.061 +0.004 -0.032 10° +1.593 +0.596 +0.274 +0.118 +0.030 -0.024 -0.056 13.49° (1500km) +1.457 +0.472 +0.159 +0.015 -0.058 -0.095 -0.110 17.99°(2000km) +1.306 +0.339 +0.049 -0.070 -0.116 -0.126 -0.118 20° +1.247 +0.290 +0.011 -0.094 -0.127 -0.125 -0.105 25° +1.133 +0.204 -0.044 -0.116 -0.120 -0.093 -0.057 30° +1.062 +0.161 -0.058 -0.103 -0.087 -0.050 -0.013 40° +1.021 +0.144 -0.051 -0.080 -0.057 -0.024 +0.002 50° +1.044 +0.122 -0.094 -0.116 **-0.0**67 -0.006 +0.033 60° +1.040 +0.024 -0.188 -0.134 +0.069 -0.009 +0.063 70° +0.958 -0.240 -0.130 -0.053 +0.095 +0.081 -0.015 80° +0.807 +0.111 -0.260 -0.187 -0.016 +0.073 -0.075 90° +0.636 -0.306 -0.068 +0.128 +0.023 -0.076 -0.011 100° +0.504 -0.274 +0.024 +0.090 -0.046 -0.034 +0.041 110° +0.441 -0.222 +0.049 +0.038 -0.043 +0.008 +0.020 120° +0.431 -0.208 +0.047 +0.028 -0.035 +0.010 +0.010 130° +0.430 -0.237 +0.078 +0.019 -0.050 +0.034 -0.003 1400 +0.389 -0.263 +0.140 -0.025 +0.050 -0.045 -0.039 150° +0.291 -0.234 +0.171 -0.107 +0.050 -0.006 -0.023 160° +0.159 -0.145 +0.127 -0.107 +0.086 -0.064 +0.044 170° +0.045 -0.044 +0.043 -0.041 +0.039 -0.037 +0.035 180° 0.000 0.000 0.000 0.000 0.000 0.000 0.000

Table 7.1.b

Coefficients	q for t	he coati	ng metho	d as fun	ctions of	f radius	Ψ
Ψ _O	d ⁵	⁹ 3	q_{14}	g_ 5	9 6	g. 7	g ₈
0°	+2.000	+1.000	+0.667	+0.500	+0.400	+0.333	+0.286
5°	+1.565	+0.697	+0.408	+0.265	+0.179	+0.123	+0.083
8.99°(1000km)	+1.226	+0.465	+0.215	+0.095	+0.026	-0.016	-0.042
10°	+1.142	+0.408	+0.170	+0.056	-0.007	-0.044	-0.066
13.49°(1500km)	+0.858	+0.222	+0.024	-0.062	-0.102	-0.119	-0.121
17.99°(2000km)	+0.512	+0.009	-0.128	-0.170	-0.173	-0.156	-0.129
20°	+0.366	-0.075	-0.182	-0.202	-0.185	-0.152	-0.113
25 °	+0.033	-0.251	-0.275	-0.233	-0.169	-0.101	-0.040
30 °	-0.255	-0.374	-0.309	-0.206	-0.103	-0.018	+0.040
40°	-0.676	-0.460	-0.228	-0.045	+0.066	+0.103	+0.085
50 °	-0.878	-0.363	-0.037	+0.116	+0.127	+0.057	-0.021
60 °	-0.875	-0.156	+0.138	+0.153	+0.040	-0.058	-0.071
70 °	-0.707	+0.070	+0.206	+0.061	-0.074	-0.073	+0.007
80°	-0.433	+0.235	+0.149	-0.064	-0.090	+0.014	+0.060
90 °	-0.121	+0.293	+0.018	-0.119	-0.007	+0.068	+0.003
100°	+0.165	+0.241	-0.101	-0.070	+0.070	+0.020	-0.050
110°	+0.376	+0.115	-0.147	+0.026	+0.060	-0.048	-0.011
120°	+0.484	-0.028	-0.105	+0.086	-0.012	-0.039	+0.040
130°	+0.488	-0.137	-0.014	+0.068	-0.058	+0.020	+0.015
140°	+0.406	-0.178	+0.065	+0.002	-0.035	+0.041	-0.029
150°	+0.275	-0.153	+0.092	-0.050	+0.018	+0.004	-0.016
160°	+0.138	-0.088	+0.066	-0.051	+0.038	-0.027	+0.018
170°	+0.037	-0.025	+0.021	-0.019	+0.017	-0.015	+0.014
180°	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Now we are going to evaluate formulas (7.26.a,b) for the direct and the coating method, for different sets of c_n . First we use the maximum estimates for c_n of [Kaula, 1959, p. 2419]:

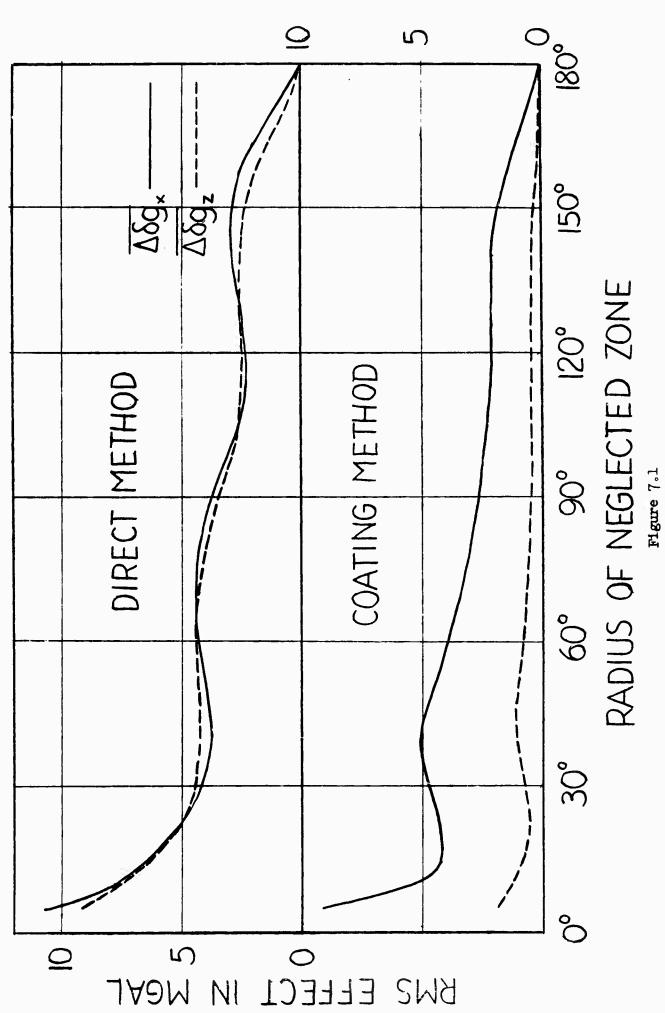
$$c_2 = 15$$
, $c_3 = 43$, $c_4 = 30$, $c_5 = c_6 = c_7 = c_8 = 25 \text{ mgal}^2$

The results are given in Table 7.2. The summation has first been extended to n = 8; higher order c_n have been neglected. In order to see the convergence of the series (7.26.a,b) we have also performed the summation up to n = 5 only.

Table 7.2. RMS influence of the zones beyond a radius Ψ_{0} on δ_{N} , δ_{M} , δ_{L} , based on the maximum estimates for the degree variances of [Kaula, 1959]

	Summation up to 8th order			Summation up to 5th order				
	Direct		Coating		Direct		Coating	
Ψ	$ abla^{arrho} \overline{\Delta}^{arrho} \underline{\Delta} abla$	$\delta_{\mathrm{M}} = \overline{\Delta} \delta_{\mathrm{L}}$	$\overline{\Delta s}_{N}$ $\overline{\Delta s}$	$\delta_{\mathrm{M}} = \Delta \delta_{\mathrm{L}}$	$\overline{\Delta}$ 8 N		$\overline{\Delta s}_{N}$ $\overline{\Delta s}$	$\delta_{\mathrm{M}} = \overline{\Delta}\delta_{\mathrm{L}}$
	mgal		mgal		mgal		mgal	
5°	9.4	10.6	2.0	9.3	9.3	10.1	2.0	8.8
8.99°(1000km)	7.8	8.1	1.4	6.0	7.8		1.4	5.9
10°	7.5		1.3	5.4	7.5		1.3	5.3
13.49°(1500km)	6.6	6.7	1.0				0.9	3.4
17.99°(2000km)	5.7	5.9	0.7	4.3		5.2	0.6	2.6
20°	5.4	5.6	0.6	4.4	5.2	4.9	0.5	2.8
25°	4.7	4.8	0.7	4.5	4.7		0.6	3.8
30°	4.4	4.1	0.8	4.7	4.3	4.0	0.8	4.6
40°	4.2	3.8	1.1	5.2	4.1	3. 7	1.0	4.8
50 °	4.2	4.0	1.1	4.6	4.2	3.9	1.0	4.3
60°	4.3	4.3	0.9	4.0	4.2	4.1	0.9	3.7
70 °	4.1	4.3	0.8	3 . 3	4.0	4.0	0.8	3.1
80°	3.8	4.2	0.6	3.1	3.7	3.8	0.6	2.8
9 0°	3.3	3.7	0.5	2.8	3.3		0.5	2.6
100°	2.7	3.0	0.5	2.6	2.7	2.9	0.5	2.3
110°	2.3	2.4	0.5	2,2	2.3	2.4	0.5	2.0
120°	2.2	2.3	0.5	2.2	2.2	2.3	0.5	2.0
130°	2.4	2.6	0.5	2,2	2.3	2.5	0.5	2.1
1400	2.4	3.0	0.5	2.2	2.4		0.5	2.0
150°	2.2	2.8	0.4	1.8	2.2	2.8	0.4	1.8
160°	1.6	2.4	0.2	1.3	1.4	2.0	0.2	1.1
170°	0.6	1.1	0.1	0.5			0.1	0.4
180°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Fig. 7.1 shows a graphical representation of the first part of Table 7.2.



Graphical representation of the values of Table 7.2 for summation up to 8th order.

One would expect a rather monotonous decrease of the curves from Oo to 180°. The non-monotonousness is especially apparent in the coating method. It is probably due mainly to neglecting terms of higher than 8th order: it is still stronger if we use only terms up to 5th order as Table 7.2. shows. It seems, however, to be hardly worth-while to go up higher than to 8th order, in view of the increasing complexity of the formulas for Q and q and the uncertainty of the numerical material for c Since the higher order terms can be expected to influence mainly the dips, it is justifiable to "bridge" them empirically in order to get a smooth monotonous function.

More recent values for the c can be obtained from [Kaula, 1961] and from [Uotila, 1962]. Both give a development of the gravity anomaly field in fully normalized spherical harmonics. Writing (7.20) in fully normalized harmonic coefficients we get [Kaula, 1959, eq. (33)]

$$c_{n} = \overline{\Delta g_{n}}^{2} = \sum_{m=0}^{n} (a_{nm}^{2} + b_{nm}^{2}).$$

By this equation, the c_n can be computed.

From [Uotila, 1962, Table 5] we get (flattening 1/297):

$$c_2 = 22$$
, $c_3 = 42$, $c_4 = 32$.

Only values up to 4th order are given. The almost perfect agreement

with the c_n of [Kaula, 1959] is accidental.

From [Kaula, 1961] we get for the flattening 1/298.24:

$$c_2 = 0.7$$
, $c_3 = 19$, $c_4 = 11$, $c_5 = 6$, $c_6 = 11$, $c_7 = c_8 = 5$;

if we use the flattening 1/297, then σ_2 is changed to $\sigma_2=17$. The influence of the distant zones, according to these values, is given in Table 7.3 and in Fig. 7.2. We see that the effect of a wrong flattening is much stronger in the direct than in the coating method. This can be expected since, as Tables 7.1.a and 7.1.b show, q_2 is smaller and changes sign more often than Q_2 .

Table 7.3.

RMS influence of the zones beyond a radius Ψ_{o} on δ_{N} , δ_{M} , δ_{L} , for two different flattenings, based on a spherical harmonics development obtained by a combined adjustment of gravimetric, astrogeodetic, and satellite data [Kaula, 1961] (summation up to 8th order)

	flattening 1/298.24			flattening 1/297				
	Direct		Coating		Direct		Coating	
¥ o	$\overline{\Delta\delta}_{N} \ \overline{\Delta\delta}_{M^{=}} \ \overline{\Delta\delta}_{L}$		$\Delta \delta_{N} \Delta \delta_{M} = \Delta \delta_{L}$		$\overline{\Delta \delta_{N}} \overline{\Delta \delta_{M}} = \overline{\Delta \delta_{L}}$		$\Delta \delta_{N} \Delta \delta_{M} = \Delta \delta_{L}$	
	mgal		mgal		mgal		mgs	al
5°	4.3	5.7	0.9	5.0	8.5	8.6	1.9	7.4
8.99°(1000km)	3.3	4.1	0.6	3.0	7.5	7.1	1.4	5.3
10°	3.1	3.8	0.5	2.6	7.2	6.8	1.3	4.8
13.49° (1500km)	2.5	3.1	0.3	1.9	6.5	6.0	0.9	3.6
17.99° (2000km)	2.0	2.5	0.3	2.2	5.7	5. 3	0.6	2.8
20°	1.8	2.4	0.3	2.4	5.4	5.0	0.5	2.7
25°	1.4	1.9	0.4	2.7	4.8	4.4	0.4	2.7
30°	1.2	1.5	0.5	2.9	4.5	4.1	0.6	3.1
40°	1.1	1.3	0.6	3.0	4.3	3.8	0.9	3.8
50°	1.2	1.4	0.5	2.4	4.4	4.0	1.0	3.9
60°	1.2	1.5	0.3	1.6	4.4	4.0	0.9	3 . 5
70 °	1.4	1.9	0.3	1.5	4.2	3.9	0.8	2.9
80°	1.6	2.1	0.3	1.8	3.6	3 . 6	0.5	2.3
90°	1.6	1.9	0.3	1.7	3.0	3.0	0.4	1.8
100°	1.3	1.7	0.3	1.6	2.4	2.4	0.3	1.7
110°	1.1	1.3	0.2	1.2	2.1	2.0	0.4	1.8
120 °	1.0	1.2	0.2	0.9	2.0	2.0	0.5	1.9
130°	1.2	1.4	0.2	1.0	2.1	2.1	0.5	2.0
140°	1.3	1.7	0.2	1.1	2.0	2.2	0.5	1.8
150 °	1.2	1.7	0.2	1.0	1.7	2.0	0.3	1.4
160 °	0.9	1.4	0.1	0.7	1.1	1.5	0.2	0.9
170 °	0.3	0.6	0.0	0.3	0.4	0.6	0.1	0.3
180°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Graphical representation of $\Delta^{\delta}_{M} = \Delta^{\delta}_{L}$ of Table 7.3.

Finally we consider the spherical harmonics expansion of Zhongolovich quoted and used in [Molodenskii et al., 1962, pp. 145-6]. Here,

$$c_2 = 61$$
, $c_3 = 96$, $c_4 = 12$, $c_5 = 8$, $c_6 = 14$, $c_7 = 5$, $c_8 = 8$ mgal

(computed by $c_n = \overline{\Delta g_n}^2$). With these coefficients we find values considerably higher than those given above (Table 7.4). They are, however, probably too high since the development of Zhongolovich is already out of date (it was computed in 1952).

Table 7.4 RMS influence of the zones beyond a radius $_{\rm o}^{\rm w}$ on $_{\rm o}^{\rm w}$, $_{\rm o}^{\rm w}$, based on the spherical harmonics development of Zhongolovich, 1952 (summation up to 8th order).

	I	Direct	Coating		
¥ o	$\overline{\mathbf{\Delta}_{\varrho}^{\mathrm{N}}}$	$ \overline{\Delta \delta}_{M} = \overline{\Delta \delta}_{L} $	$\overline{\Delta \varepsilon}_{ extbf{N}}$	$ \underline{\nabla}_{\mathbf{W}^{\pm}} \nabla_{\mathbf{V}} $	
-		gal	mg	al	
.5 °	16.2	16.0	3.5	13.9	
8.99°(1000km)	14.2	13.6	2.6	10.1	
10°	13.8	13.0	2.4	9.2	
13.49°(1500km)	12.3	11.5	1.8	6.6	
~17.99°(2000km)	10.7	9•9	1.0	4.2	
20°	10.2	9.3	8.0	3.7	
25°	9.1	8.2	0.7	4.0	
30°	8.4	7.5	1.1	5.3	
40°	8.1	7.2	1.7	7.4	
50 °	8.2	7.3	1.9	7∘5	
60°	8.2	7.2	1.8	6.4	
70 °	7.6	6.9	1.4	5.0	
80°	6.8	6.5	1.0	4.3	
90 °	5.8	5. 7	0.8	3.7	
100°	4.8	4.8	0.7	3.3	
1 1 0°	4.1	4.0	0.8	3.4	
120°	4.0	3.9	1.0	3.4	
1 3 0°	4.1	4.1	1.0	3.7	
140°	4.0	4.2	0.9	3 . 5	
150 °	3.3	3. 6	0.7	2.7	
1 6 0°	2.0	2.4	0.4	1.5	
170 °	0.6	0.8	0.1	0.5	
180°	0.0	0.0	0.0	0.0	

Conclusion. On the whole, the values of Fig. la,b, based on [Kaula, 1959] and confirmed by [Uotila, 1962], might be the most realistic estimates, apart from the dips. We should like to propose the smoothed values of Table 7.5.

Table 7.5 RMS influence of the zones beyond a radius Ψ_{0} on $\delta_{N},~\delta_{M},~\delta_{L};$ proposed values.

		Direct		Coating	
	Ψ o	$\overline{\Delta\delta}_{ m N}$ mgs		$\overline{\Delta g}^{N}$	$\overline{\Delta\delta}_{M}^{=} \overline{\Delta\delta}_{L}^{=}$
8.99°	(1000km)	8	8	2	6
13.49°	(1500km)	7	7	2	5
17.99°	(2000km)	6	6	2	5
	20°	6	6	2	5
	25°	5	5	2	5
	30°	5	5	1	5
	45°	5	5	1	5
	60°	4	5	ı	14
	90°	3	14	1	3
:	120 °	2	3	1	2
:	150°	2	3	0	2

These values are referring to a flattening in the neighborhood of 1/297. If we use a better flattening, then these values should be somewhat less.

We see that for $\Psi_0 > 20^\circ$ or 30^\bullet the influence of the distant zones decreases very slowly. It is, therefore, impractical to extend the integration much farther than 20° (coating method) or 30° (direct method), unless it is extended over the whole earth.

The effect of the distant zones on δ_M and δ_L is somewhat less in the coating than in the direct method. The effect on δ_N is much less. By using another method which will be described in the next section, however, the influence of the distant zones on δ_N can be made still smaller.

8. PRACTICAL COMPUTATION OF THE GRAVITY DISTURBANCES

We shall describe now a method for the practical computation of gravity disturbances. Since the geoid undulations N have been computed for large parts of the world--the accuracy is as good as the existing gravity material--we can assume them to be known. Therefore, the coating method can be applied which has certain advantages over the direct method: simplicity, less influence of the distant zones, etc.

Upward Continuation Method. For the vertical component δ there is, however, a method which is still better than the coating method.

Compute the vertical component at ground (or at the geoid) by

(8.1)
$$\delta_n^{\circ} = \Delta g + \frac{2\gamma}{R} N$$
.

Since $r\delta_n = -r \frac{\partial T}{\partial r}$ is a harmonic function, Poisson's integral for harmonic functions can be applied to give

$$r\delta_{n} = \frac{r^{2} - R^{2}}{4\pi R} \int \int \frac{R\delta_{n}^{\circ}}{D_{s}^{3}} \cdot R^{2} d\sigma$$

where

$$r = R + H$$
,
 $D_{s} = \sqrt{R^{2} + r^{2} - 2Rr \cos \Psi}$.

Setting as before

$$t = \frac{R}{r} = \frac{R}{R + H},$$

$$D = \sqrt{1 - 2t \cos \Psi + t^2}$$

we have

$$D_s = rD$$
 .

Hence

(8.2)
$$\delta_{n} = \frac{t^{3}(1-t^{3})}{4\pi} \iint_{\sigma} \frac{\delta_{n}^{\circ}}{D^{3}} d\sigma$$

where δ_n° is given by (8.1).

This formula (8.2) is called "upward continuation integral"; it can also be used for the upward continuation of the gravity anomalies Δg , since $r\Delta g$ is also a harmonic function. Here,

however, it is used for the upward continuation of the gravity disturbances.

If we compare (8.2) to the corresponding formula for the coating method,

(8.3)
$$\delta_{n} = \frac{1}{2\pi} \iint_{\sigma} \mu \cdot \frac{t^{2}}{D^{3}} (1 - t \cos \Psi) d\sigma$$

where

$$\mu = \Delta g + \frac{3\gamma}{2R} N ,$$

then we find that the integrand in (8.3) decreases, with increasing distance D , like 1/D, whereas in the upward continuation integral (8.2) it decreases much faster, like $1/D^3$. It can, therefore, be expected that the influence of the distant zones in the upward continuation method is much smaller than in the coating method.

Indeed, if we put H=0, then $\delta_n=\delta_n^o$, i.e., the gravity disturbance is equal to the point value in P itself and it does not at all depend on the other values and on the distant zones. Thus, for H=0 the influence of the distant zones is rigorously zero in the upward continuation method whereas, as we have seen in section 7, this is not at all the case in the coating method (neither in the direct method, as a matter of fact). From this we can conclude that, even for higher elevations, only the nearest surroundings of P will be of some influence.

Therefore, we can replace the terrestrial sphere by its tangential plane at P. If in this plane we assume a rectangular coordinate system x,y (origin P; the x-axis pointing northward, the y-axis pointing eastward), we have instead of (8.2) the plane formula

(8.4)
$$\delta_{n} = \frac{H}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta_{n}}{(H^{2} + x^{2} + y^{2})^{3/2}} dx dy$$
.

In order to estimate the influence of the distant zones beyond a certain distance s from P we introduce polar coordinates s, α by

$$x = s \cos \alpha$$
,

$$y = s \sin \alpha$$
,

getting

$$\delta_{n} = \frac{H}{2\pi} \int_{Q-0}^{2\pi} \int_{s-0}^{\infty} \frac{\delta_{n}}{(H^{2} + s^{2})^{3/2}} s ds d\alpha$$
.

The influence $\Delta \delta_n$ of the zones beyond the distance s is given by

$$\Delta \delta_{n} = \frac{H}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=s_{0}}^{\infty} \frac{\delta_{n}^{\circ}}{(H^{2}+s^{2})^{3/2}} s ds d\alpha.$$

Since for large s, $\sqrt{H^2 + s^2} \stackrel{!}{=} s$, we can simplify this to

$$\Delta \delta_{n} = H \int_{s}^{\infty} \frac{\delta_{n} s}{s^{\frac{2}{3}}} ds$$

where $\delta_{n,s}^{\bullet}$ is the average of δ_{n}^{\bullet} over the circle with radius s.

If M is the upper limit of the absolute amount of δ , s,

$$\delta_{n,s}^{\circ} < M$$
,

we get

$$\Delta \delta_n < HM \int_{s_0}^{\infty} \frac{ds}{s^2} = \frac{H}{s_0} M$$
.

To get some kind of average effect $\overline{\Delta\delta}_n$ rather than this maximum estimate, we may replace M by the average disturbance \overline{M} outside the radius so which is certainly close to 0:

$$(8.5) \qquad \overline{\Delta \delta_n} = \frac{H}{s} \overline{M} .$$

From this we see immediately that, to get the same error for different elevations, s must be proportional to H. If, e.g., s = 10 H, then $\overline{\Delta \delta_n} = 0.1 \ \overline{M} \ .$

Since M will hardly exceed 10 mgal, $\overline{\Delta\delta}_n$, in this case, will be smaller than 1 mgal. So, it should, in general, be sufficient to go as far as 10 times the elevation. For larger H, conditions are even more favorable since, the larger s is, the smaller can \overline{M} be expected to be.

Plane formulas analogous to (8.4) also hold for the horizontal components δ_m and δ_ℓ :

(8.6)
$$\left\{ \begin{array}{l} \delta_{\mathrm{m}} \\ \delta_{\ell} \end{array} \right\} = \frac{\mathrm{H}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\mathrm{H}^{3} + \mathrm{x}^{3} + \mathrm{y}^{3})^{3/2}} \left\{ \begin{array}{l} \delta_{\mathrm{m}} \\ \delta_{\ell} \end{array} \right\} \, \mathrm{dx} \, \mathrm{dy} .$$

But, since $\delta_m^o = \gamma \xi$, $\delta_\ell^o = \gamma \eta$, the use of these formulas would require the knowledge of the components ξ , η of the deflection of the vertical in the entire neighborhood of P . If we know ξ and η , then

these formulas can indeed be applied with advantage. (To compute § and η by differentiating the N-field will not, in general, be accurate enough.) In the following, however, we shall regard only Δg and N as known, so that δ_m and δ_ℓ must be computed by the coating method.

Formulas and Approximations. For easy reference we put the relevant formulas together:

(8.7.a)
$$\delta_n = \frac{t^2(1-t^2)}{4\pi} \iint_{\Omega} \frac{\delta_n^{\circ}}{D^3} d\sigma$$
,

(8.7.b)
$$\begin{cases} \delta_{\rm m} \\ \delta_{\ell} \end{cases} = \frac{t^3}{2\pi} \int_{\sigma} \frac{\mu}{D^3} \sin \Psi \begin{cases} \cos \alpha \\ \sin \alpha \end{cases} d\sigma$$

where

$$t = \frac{R}{R+H}, \quad D = \sqrt{1 - 2t \cos \Psi + t^2}, \quad d\sigma = \sin \Psi \ d\Psi \ d\alpha;$$

$$\delta_n^{\circ} = \Delta g + \frac{2\gamma}{R} N,$$

$$\mu = \Delta g + \frac{3\gamma}{2R} N.$$

At least for the nearest neighborhood of P, the spherical formulas can be approximated by plane ones:

(8.8.a)
$$\delta_n = \frac{H}{2\pi} \iint \frac{\delta_n^{\circ}}{D_n^{\circ}} dx dy$$
,

$$(8.8.b) \quad \begin{cases} \delta_{\rm m} \\ \delta_{\ell} \end{cases} = \frac{1}{2\pi} \quad \iint \frac{\mu}{D_{\rm o}^3} \begin{cases} x \\ y \end{cases} \, \mathrm{d}x \, \, \mathrm{d}y$$

where

(8.9)
$$D_0 = \sqrt{H^2 + x^2 + y^2}$$
.

They follow from (8.7.a,b) by a series development with respect to $\Psi=\frac{s}{R}$ and to $\frac{H}{R}$ if we put $s\cos\alpha=x$, $s\sin\alpha=y$.

In order to get an estimate on the validity of these approximations, we develop $1/\mathbb{D}^3$ or, which is equivalent,

$$\frac{1}{D^3} = \frac{1}{(R+H)^3 D^3} = \frac{1}{(R^2 + r^2 - 2Rr \cos \Psi)^{3/2}}$$

in this manner, finding

(8.10)
$$\frac{1}{D_s^3} = \frac{1}{D_o^3} \left(1 - \frac{3}{2} \frac{H}{R} \frac{s^2}{D_o^2} + \frac{1}{8} \frac{s^4}{R^2 D_o^2} + \dots \right).$$

Here $1/D_s^3$ is spherical and $1/D_o^3$ is the equivalent plane quantity (8.9); and $s^2 = x^2 + y^2$.

The first small quantity in the bracket is

$$\frac{3}{2} \frac{H}{R} \frac{s^2}{D^2} \approx \frac{3H}{2R} ,$$

independent of the distances. If H=63.7 km, this is 0.015 or 1.5%, if H=637 km it is 15%. Since the higher up we go the smaller the gravity disturbances are, this can be neglected anyway. The second term is

$$\frac{s^4}{8R^2D_o^2} \approx \frac{s^2}{8R^2}.$$

For s = 1000 km this is 0.003 or 0.3%; for s = 2000 km it is 0.01 or 1%. This can be neglected, too; the more so as $1/D_0^3$ itself becomes smaller with increasing distance.

We see that up to 20° distance from P we can, in general, use the plane formulas (8.8.a,b) instead of the spherical ones (8.7.a.b). In particular this holds practically always for the upward continuation integral (8.8.a), except for very high elevations (> 250 km, say). It is also useful in the case that in the coating formulas (8.7.b) we extend the integration only up to 20° (as we have seen, it is impractical to go much farther unless the integration is extended over the whole earth, which of course is necessary for highest accuracy).

Now we have to show how the x,y in (8.8.a,b) are to be computed. The simplest way is

another way,

(8.12)
$$x = R(\phi - \phi_p) ,$$

$$y = R \cos \phi (\lambda - \lambda_p) .$$

The x's are the same in both; the y's differ in that in (8.11) we have the factor $\cos\phi_P$ and in (8.12), the factor $\cos\phi$. In our latitudes there is, for large $|\phi-\phi_P|$, a big difference between both formulas. Take $\phi_P=40^{\circ}$, $|\phi-\phi_P|=20^{\circ}$, i.e., $\phi=20^{\circ}$ or 60° . Then

$$\cos 20^{\circ} = 0.940$$
,

$$\cos 40^{\circ} = 0.766$$
 ,

$$\cos 60^{\circ} = 0.500$$
 .

The differences between cos ϕ and cos ϕ_p are 22% and 35%:

It is easily seen that (8.12) is preferable for larger distances; only if we use this formula rather than (8.11) is the reasoning following (8.10) applicable. The disadvantage of (8.12) is that the 5'x 5', 1°x 1°, etc., blocks are not rectangles in the system x,y but trapezoids. In (8.11) these blocks are represented by rectangles. Evaluation of the Integrals. In practice, the integrals must be evaluated by sums. The formulas (8.7.a,b) and 8.8.a,b) are of the type (8.13.a) $\delta = \iint \mu \cdot f(\varphi,\lambda) dq$,

(8.13.b)
$$\delta = \iint \mu \cdot f(x,y) dq$$
.

In δ we have δ instead of μ ; Ψ,α,D in (8.7.a,b) are, of course, functions of the geographical coordinates ϕ , λ ; dq is the element of area, defined by

$$dq = R^{2}d\sigma = R^{2}\cos \phi d\phi d\lambda ,$$

$$dq = dx dy,$$

respectively.

The preceding integrals can be approximated by sums:

$$(8.14) \qquad \delta = \Sigma \ \mu_{k} \ c_{k}$$

where

(8.15.a)
$$c_k = f(\varphi_k, \lambda_k) \cdot q_k$$
, or

(8.15.b)
$$c_k = f(x_k, y_k) \cdot q_k$$

where q_k is the area of a certain block B_k (5'×5' or 1°×1°, say), and ϕ_k , λ_k or x_k , y_k are the coordinates of the center of the block B_k .

Especially for low elevations and for the nearest neighborhood of P it is better to compute $\,c_k^{}\,$ in the <code>integrated form.</code>

We limit ourselves to rectangular coordinates x, y. Then

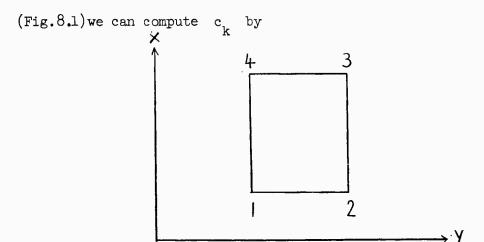


Figure 8.1

(8.16)
$$c_k = F(1) - F(2) + F(3) - F(4)$$

where

$$(8.17) F(x,y) = \iint f(x,y) dx dy$$

is the indefinite double integral of f(x,y) so that

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

and, e.g., F(1) is the value of F(x,y) in point 1.

As a matter of fact, this is equivalent to computing c_k by

(8.18.a)
$$c_{k} = \iint_{B_{k}} f(x,y) dx dy$$

rather than by

$$(8.18.b)$$
 $c_k = f(x_k, y_k) \cdot q_k$.

If f(x,y) varies little in the block B_k considered, then (8.18.a) and (8.18.b) are practically equal. If not, then (8.18.a), i.e., (8.16), is to be preferred: if μ is constant throughout the block, then (8.18.a) yields the correct result whereas (8.18.b) does not. (8.18.b) is in this case, therefore, subject to a systematical error.

Let us now return to formulas (8.8.a,b). For (8.8.a),

$$f_n(x,y) = \frac{H}{2\pi D_0^3} = \frac{H}{2\pi (x^3 + y^2 + H^3)^{3/2}};$$

for (8.8.b),

$$f_{m}(x,y) = \frac{x}{2\pi D_{o}^{3}} = \frac{x}{2\pi (x^{2}+y^{2}+H^{2})^{3/2}}$$
,

$$f_{\ell}(x,y) = \frac{y}{2\pi D_{0}^{3}} = \frac{y}{2\pi (x^{3}+y^{3}+H^{2})^{3/2}}$$
.

Integration by (8.17) yields

(8.19.a)
$$F_n(x,y) = \frac{1}{2\pi} \tan^{(-1)} \frac{xy}{HD}$$
,

(8.19.b)
$$F_{m}(x,y) = -\frac{1}{2\pi} \ln (y + D_{o})$$
,

(8.19.c)
$$F_{\ell}(x,y) = -\frac{1}{2\pi} \ln (x + D_0)$$
.

Then, the coefficients c_k for δ_n , δ_m , δ_k can be computed by (8.16).

In order to be able to apply (8.16), the figure 1234 must be an exact rectangle. Therefore, x,y must in this case be computed by (8.11) instead of (8.12). Therefore, we are limited to the nearest surroundings of P where (8.11) and (8.12) are practically equal, say to a rectangle $3^{\circ} \times 4^{\circ}$ in the center of which P is situated (Region A, p. 25). Outside this rectangle, c_k can very well be computed by the simpler formula (8.18.b).

For the computation of the contribution of the innermost zones it is necessary to use as small blocks as possible. The smallest size is usually 5 ' × 5 '. Even so, it might be necessary to take the deviation of the gravity anomalies from the mean anomaly of the respective block into account.

For this purpose we consider the gravity anomaly to be a linear function throughout the block. Then,

(8.20)
$$\Delta g = a_0 + a_1 x + a_2 y$$
.

Inserting this in (8.8.a) and (8.8.b) and integrating over one block B_k only we get

$$\Delta \delta_{n} = H(a_{0}J_{0} + a_{1}J_{1} + a_{2}J_{2}) ,$$

$$\Delta \delta_{m} = a_{0}J_{1} + a_{1}J_{11} + a_{2}J_{12} ,$$

$$\Delta \delta_{\ell} = a_{0}J_{2} + a_{1}J_{12} + a_{2}J_{22} ,$$

where $\Delta\delta_n$, $\Delta\delta_m$, $\Delta\delta_k$ are the contributions of the block B_k to δ_n , δ_m , δ_k and

$$(8.21) J = F(1) - F(2) + F(3) - F(4)$$

where

$$F_{o}(x,y) = \frac{1}{2\pi} \iint \frac{dx \, dy}{D_{o}^{3}} = \frac{1}{2\pi H} \tan^{(-1)} \frac{xy}{HD_{o}} = \frac{1}{H} F_{n}(x,y) ,$$

$$F_{1}(x,y) = \frac{1}{2\pi} \iint \frac{x \, dx \, dy}{D_{o}^{3}} = -\frac{1}{2\pi} \ln(y+D_{o}) = F_{m}(x,y) ,$$

$$F_{2}(x,y) = \frac{1}{2\pi} \iint \frac{y \, dx \, dy}{D_{o}^{3}} = -\frac{1}{2\pi} \ln(x+D_{o}) = F_{\ell}(x,y) ,$$

$$(8.22)$$

$$F_{11}(x,y) = \frac{1}{2\pi} \iint \frac{x^{3} \, dx \, dy}{D_{o}^{3}} = \frac{y}{2\pi} \ln(x+D_{o}) - \frac{H}{2\pi} \tan^{(-1)} \frac{xy}{HD_{o}} ,$$

$$F_{12}(x,y) = \frac{1}{2\pi} \iint \frac{xy \, dx \, dy}{D_{o}^{3}} = -\frac{D_{o}}{2\pi} ,$$

$$F_{22}(x,y) = \frac{1}{2\pi} \iint \frac{y^{2} \, dx \, dy}{D_{o}^{3}} = \frac{x}{2\pi} \ln(y+D_{o}) - \frac{H}{2\pi} \tan^{(-1)} \frac{xy}{HD_{o}} .$$

Consider now the contribution of the innermost block. Let the sides of the block be 2a, 2b, so that the points 1, 2, 3, 4 have the coordinates:

Then,

$$J_1 = J_2 = J_{12} = 0$$

and

$$(8.23.a)$$
 $\Delta\delta_n = a HJ$,

(8.23.b)
$$\Delta \delta_{m} = a_{1} J_{11}$$
,

(8.23.c)
$$\Delta \delta_{\ell} = a_{2} T_{22}$$
,

where

$$J_{0} = \frac{2}{\pi H} \tan^{(-1)} \frac{ab}{HD_{0,1}},$$

$$J_{11} = \frac{b}{\pi} \ln \frac{D_{0,1} + a}{D_{0,1} - a} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_{0,1}},$$

$$(8.24)$$

$$J_{22} = \frac{a}{\pi} \ln \frac{D_{0,1} + b}{D_{0,1} - b} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_{0,1}};$$

$$D_{0,1} = \sqrt{a^{2} + b^{2} + H^{2}},$$

We see that for the central block $\Delta \delta_n$ is the same as if we would use formula (8.16). For $\Delta \delta_m$ and $\Delta \delta_n$ (8.16) yields zero whereas the correct values are given by (8.23.b,c).

In order to estimate numerically the contribution of the innermost compartment, $\Delta\delta_m$ and $\Delta\delta_\ell$, we take it as a square with sides s , so that

$$a = b = s ; D_{0,1} = \sqrt{2s^2 + H^2}.$$

Since, then, $J_{\text{ll}} = J_{\text{22}}$ it is sufficient to consider $\Delta \delta_{\text{m}}$ only.

We have

(8.25)
$$J_{11} = \frac{s}{\pi} \ln \frac{D_{0,1} + s}{D_{0,1} - s} - \frac{2H}{\pi} \tan^{(-1)} \frac{s^2}{HD_{0,1}};$$

we take

$$s = 4 \text{ km}$$

which corresponds to the average size of 5 \times 5 blocks. Table 8.1 gives the values of J_{11} for several elevations H .

Н	J ₁₁		
km	km		
0	2.24		
1	1.41		
2	0.87		
5	0.23		
10	0.04		
20	0.01		
50	0.00		
100	0.00		

If we have a north-south gradient a_1 = 1 mgal per 1 km, then $J_{11} \mbox{ is numerically equal to the effect } \Delta \delta_m \mbox{ in mgal; e.g., for H = 0}$ we have

$$\Delta \delta_{\rm m} = 2.2 \, {\rm mgal}$$
 .

Since also gradients of 2 or even 3 mgal/km are possible, the effect of the gradient on the horizontal components $\Delta \delta_{\rm m}$ and $\Delta \delta_{\ell}$ can be considerable for lower elevations, H < 5 km. For H > 5 km, the effect of the gradient is negligible.

We have also to investigate the influence of the eight blocks adjacent to the central one. Since the effect is a maximum for H=0, we consider this case only. We again take the block to be a square, but with arbitrary sides. We find that the effect of the innermost block is

(8.26)
$$\Delta \delta_{\rm m} = 0.561 \ a_1 s$$

and the combined effect of the eight surrounding blocks is only

(8.27)
$$\Delta \delta_{m} = -0.071 a_{1}s$$
.

If the gradient $a_1 = 1 \text{ mgal/km}$ and s = 4 km (5 x 5 blocks), then

$$\Delta \!\!\!\!/ \, \delta_m = -0.3 \text{ mgal}$$
 .

These values hold for H=0, but even here the effect of the gradient in the surrounding blocks will be negligible in most cases, even if it must be taken into account for the central block (for H < 5 km).

These considerations are equally valid for the component δ_{ℓ} ; we have only to replace a_1 by the east-west gradient a_2 . For the vertical component δ_n , the influence of the gradients a_1 and a_2 will always be negligible.

An easy way of computing the gradients a_1 and a_2 which can also be used for high-speed computations is (Fig. 8.2):

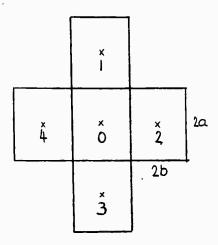


Figure 8.2 Station and Substations

(8.28)
$$a_1 = \frac{\Delta g_1 - \Delta g_3}{4a}$$
, $a_2 = \frac{\Delta g_2 - \Delta g_4}{4b}$.

 Δg_1 through Δg_4 are the gravity anomalies in the centers of the adjacent blocks 1 to 4, which we call substations. These point anomalies can also be replaced by the mean anomalies of the respective blocks, provided the gravity anomalies are sufficiently linear (this is presupposed in (8.28)).

Station and Substations. For reasons of symmetry the station P at which δ_n , δ_n , δ_n , are computed must be at the center of the central block. If we, however, use given 5 \times 5 mean anomalies for which the division into blocks is fixed a priori, it is best to select 5 substations (centers of adjacent blocks) 0, 1, 2, 3, 4 (Fig. 8.2)

and compute the gravity disturbance in those three substations which are the corners of a triangle containing P in its interior (0, 1, 4 in Fig. 8.2). The value for P is then interpolated between these three substations. This procedure is particularly necessary for elevations smaller than about 70 km.

Summary of Formulas and Outline of Computation. In the following
we shall give a computational method valid for any elevation from

O to several hundred kilometers. It is, therefore, suitable for the
computation of the gravity vector along rocket trajectories.

According to p.25 we use the following regions

A
$$\varphi_2 - \varphi_1 = 3^{\circ}$$
, 5 x 5 blocks, $\lambda_2 - \lambda_1 = 4^{\circ}$

B
$$\varphi_2 - \varphi_1 = 7^\circ$$
 outside A, 20 × 20 blocks, $\lambda_2 - \lambda_1 = 9^\circ$

C
$$\phi_{2} - \phi_{1} = 25^{\circ}$$
 outside B, 1° × 1° blocks,
$$\lambda_{2} - \lambda_{1} = 30^{\circ} \dots$$

D outside C, $5^{\circ} \times 5^{\circ}$ blocks.

1. For each block compute

$$\mu = \Delta g + 0.231 \text{ N}$$
, $\delta_n^{\circ} = \Delta g + 0.308 \text{ N}$;

 Δg : mean free-air anomalies, N: geoid undulations; Δg , μ , δ_n° in mgal, N in meters.

2. For each station P select 3 substations, according to Fig. 8.2.

One substation, O, is the center of the block in which P lies, the other two are centers of adjacent blocks so that the three substations form the corners of a triangle containing P. The elevation of each substation is equal to the elevation H of P.

The gravity disturbances are computed at each substation and are interpolated for P .

In the following, only the substations will be considered; they will be denoted by Q.

3. Now we consider one definite substation Q only.

Zone A.

Compute for all grid points (= corners of blocks):

$$x = R(\phi - \phi_Q),$$

$$y = R \cos \phi_Q (\lambda - \lambda_Q);$$

$$D = \sqrt{x^2 + y^2 + H^2};$$

$$F_{n} = \frac{1}{2\pi} \tan^{(-1)} \frac{xy}{hD_{o}},$$

$$F_{m} = -\frac{1}{2\pi} \ln (y + D),$$

$$F_{\ell} = -\frac{1}{2\pi} \ln (x + D).$$

Compute for each block

$$c_{n} = F_{n}(1) - F_{n}(2) + F_{n}(3) - F_{n}(4) ,$$

$$c_{m} = F_{m}(1) - F_{m}(2) + F_{m}(3) - F_{m}(4) ,$$

$$c_{\ell} = F_{\ell}(1) - F_{\ell}(2) + F_{\ell}(3) - F_{\ell}(4) .$$

Be sure that the grid points 1, 2, 3, 4 are arranged as in Fig. 8.1.

For elevations $H > 40 \ km$ the coefficients c can also be computed as in zone B.

Zone B

Compute for all block centers

$$x = R(\phi - \phi_{Q}),$$

$$y = R \cos \phi(\lambda - \lambda_{Q});$$

$$D = \sqrt{x^{2} + y^{2} + H^{2}};$$

$$c_{n} = \frac{H}{2\pi D_{Q}^{3}} \cdot q,$$

$$c_{m} = \frac{x}{2\pi D_{Q}^{3}} \cdot q;$$

$$c_{\ell} = \frac{y}{2\pi D_{Q}^{3}} \cdot q;$$

q is the area of the block.

Zone C can be treated either like Zone B or like Zone D.

Zone D

If many points are to be computed then the following computations need not be performed for all points. In this case it is sufficient to compute the effect of Zone D for a few points only and interpolate between these.

Compute:

$$\begin{split} \Psi &= \cos^{\left(-1\right)} \left[\sin \phi_{Q} \sin \phi + \cos \phi_{Q} \cos \phi \cos \left(\lambda - \lambda_{Q}\right) \right] \; , \\ \alpha &= \cot^{\left(-1\right)} \left[\frac{\cos \phi_{Q} \sin \phi - \sin \phi_{Q} \cos \phi \cos \left(\lambda - \lambda_{Q}\right)}{\cos \phi \sin \left(\lambda - \lambda_{Q}\right)} \right] \; ; \\ t &= \frac{R}{R + H} \; , \\ D &= \sqrt{1 - 2t \cos \Psi + t^{2}} \; ; \\ c_{n} &= \begin{cases} 0 \; , \; H \leq 200 \; km \\ \frac{t^{2} \left(1 - t^{2}\right)}{4\pi D^{3}} \; . \; q \; , \; H > 200 \; km \end{cases} \; , \\ c_{m} &= \frac{t^{3}}{2\pi D^{3}} \sin \Psi \cos \alpha \cdot q \; , \\ c_{L} &= \frac{t^{3}}{2\pi D^{3}} \sin \Psi \sin \alpha \cdot q \; . \end{split}$$

4. For the substation Q considered compute by summation over all blocks i:

$$\delta_{n} = \sum_{i} c_{n,i} \delta_{n,i}^{\circ},$$

$$\delta_{m} = \sum_{i} c_{m,i} \mu_{i},$$

$$\delta_{\ell} = \sum_{i} c_{\ell,i} \mu_{i}.$$

For elevations < 5 km we have to add, to $\delta_{\rm m}$ and to δ_{ℓ} , the terms $\Delta\delta_{\rm m}$ and $\Delta\delta_{\ell}$, respectively:

$$\Delta \delta_{m} = a_{1} \cdot \left[\frac{b}{\pi} \ln \frac{D_{1}+a}{D_{1}-a} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_{1}} \right],$$

$$\Delta \delta_{\ell} = a_{2} \cdot \left[\frac{a}{\pi} \ln \frac{D_{1}+b}{D_{1}-b} - \frac{2H}{\pi} \tan^{(-1)} \frac{ab}{HD_{1}} \right],$$

where 2a,2b are the sides of the central 5' x 5' block and

$$D_{1} = \sqrt{a^{2} + b^{2} + H^{2}};$$

$$a_{1} = \frac{\Delta g_{1} - \Delta g_{3}}{4a}, \quad a_{2} = \frac{\Delta g_{2} - \Delta g_{4}}{4b}.$$

These formulas for a_1 and a_2 are valid for the central substation 0 (Fig. 8.2); for the other substations they are to be modified in an evident way.

5. Now denoting by x,y,z the coordinates in the geocentric system defined on p. 2 (z-axis = axis of rotation of the earth) compute the components of the gravity disturbance in this system by

$$\delta_{\mathbf{x}} = -\cos \varphi \cos \lambda \cdot \delta_{\mathbf{n}} - \sin \varphi \cos \lambda \cdot \delta_{\mathbf{m}} - \sin \lambda \cdot \delta_{\ell},$$

$$\delta_{\mathbf{y}} = -\cos \varphi \sin \lambda \cdot \delta_{\mathbf{n}} - \sin \varphi \sin \lambda \cdot \delta_{\mathbf{m}} + \cos \lambda \cdot \delta_{\ell},$$

$$\delta_{\mathbf{z}} = -\sin \varphi \cdot \delta_{\mathbf{n}} + \cos \varphi \cdot \delta_{\mathbf{m}}.$$

These components δ_x , δ_y , δ_z are positive <u>away from</u> the earth's center; ϕ , λ refer to the substation Q.

6. This procedure must be repeated for all substations and the values of δ_x , δ_y , δ_z for the station P are interpolated.

For elevations H > 70 km, it is in general sufficient to use the central substation 0 only and to put the values at P equal to the values at 0.

7. Computation of the components of the gravity vector:

$$g_{x} = Y_{x} - \delta_{x},$$

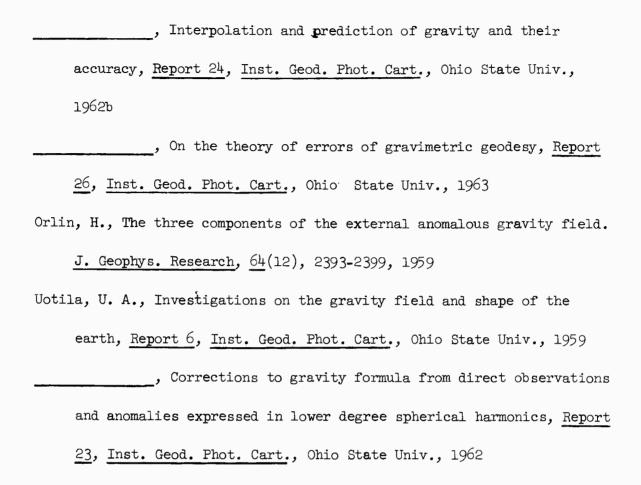
$$g_{y} = Y_{y} - \delta_{y},$$

$$g_{z} = Y_{z} - \delta_{z}.$$

 g_x , g_y , g_z ; γ_x , γ_y , γ_z are positive toward the earth's center.

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PRACTICAL COMPUTATION OF GRAVITY AT HIGH ALTITUDES. TR 1613-1, May 63, Bedford, Mass. Rpt. No. AFCRL-63-702. AF Cambridge Research Laboratories,

gravity. For gravity disturbances, different distant zones, etc.); finally a practical comgravity disturbance. The report first gives means of accuracy studies (on influence of gravity is split up into normal gravity and a practical method for computing normal Methods for computing the gravity vector outside the earth are presented and commethods are described and compared by pared. For the purpose of computation, putation procedure is given.

88 p. incl illus., tables and 12 refs.

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Geodetic Computations

Gravity

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Error Analysis

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